Part II

Diagnostic Equations

Chapter 4

Equation of State

The *in situ* density is needed to calculate the pressure gradient term in the momentum equation. As indicated in (2.5), the equation of state represents *in situ* density as a function of pressure, temperature, and salinity. Here we present the specific form of the equation of state. We still adhere to the 1980 International Equation of State of Seawater (EOS-80; UNESCO, 1981). Thermodynamic Equation Of Seawater - 2010 (TEOS-10; IOC, SCOR, and IAPSO, 2010) has not been adopted.

4.1 Basics of the equation of state

The standard equation of state provided by UNESCO (1981) represents density (kg m^{-3}) as a function of *in situ* temperature* t (°C), salinity S (practical salinity scale (pss) ~ parts per thousand (ppt)), and pressure P (bar). Note that *in situ* temperature is used, not potential temperature. Density (ρ_w) of pure water ($S = 0$) under sea level pressure is given as a function of temperature (t):

$$\begin{aligned} \rho_w(t) &= 999.842594 + 6.793952 \times 10^{-2} t - 9.095290 \times 10^{-3} t^2 \\ &\quad + 1.001685 \times 10^{-4} t^3 - 1.120083 \times 10^{-6} t^4 + 6.536332 \times 10^{-9} t^5. \end{aligned} \tag{4.1}$$

Density at the sea surface ($\rho_{\text{surf}} = \rho(t, S, 0)$) is expressed using sea surface temperature and salinity:

$$\begin{aligned} \rho_{\text{surf}} &= \rho_w \\ &\quad + (0.824493 - 4.0899 \times 10^{-3} t + 7.6438 \times 10^{-5} t^2 - 8.2467 \times 10^{-7} t^3 + 5.3875 \times 10^{-9} t^4) S \\ &\quad + (-5.72466 \times 10^{-3} + 1.0227 \times 10^{-4} t - 1.6546 \times 10^{-6} t^2)\, S^{\frac{3}{2}} \\ &\quad + 4.8314 \times 10^{-4} S^2. \end{aligned} \tag{4.2}$$

The *in situ* density ($\rho = \rho(t, S, P)$) is converted from ρ_{surf} using the following equation,

$$\rho = \rho_{\text{surf}} / (1 - P/K), \tag{4.3}$$

where $K(S, t, P)$ is the secant bulk modulus. Its value at pure water, K_w, is given by

$$K_w = 19652.21 + 1.484206 \times 10^2 t - 2.327105 t^2 + 1.360477 \times 10^{-2} t^3 - 5.155288 \times 10^{-5} t^4. \tag{4.4}$$

The value at the sea surface (K_0) is given by

$$\begin{aligned} K_0 &= K_w + (54.6746 - 0.603459 t + 1.09987 \times 10^{-2} t^2 - 6.1670 \times 10^{-5} t^3) S \\ &\quad + (7.944 \times 10^{-2} + 1.6483 \times 10^{-2} t - 5.3009 \times 10^{-4} t^2)\, S^{\frac{3}{2}}, \end{aligned} \tag{4.5}$$

* EOS-80 provided by UNESCO (1981) was based on temperature on the International Practical Temperature Scale of 1968 (IPTS-68; t_{68}) and the Practical Salinity Scale 1978 (PSS-78). After that the International Committee for Weights and Measures adopted a new temperature scale (the International Temperature Scale of 1990 (ITS-90); t_{90}). It has been recommended that t_{90} should be converted to t_{68} by a relation $t_{68} = 1.00024 * t_{90}$ when it is used for EOS-80 (Saunders, 1990). However, this conversion has not been applied for temperature in any operation of MRI.COM.

Table4.1 Coefficients for the lapse rate of sea water (Eq. 4.8).

a_0	$+3.5803\times10^{-5}$	b_0	$+1.8932\times10^{-6}$	c_0	$+1.8741\times10^{-8}$	d_0	-1.1351×10^{-10}
a_1	$+8.5258\times10^{-6}$	b_1	-4.2393×10^{-8}	c_1	-6.7795×10^{-10}	d_1	$+2.7759\times10^{-12}$
a_2	-6.8360×10^{-8}			c_2	$+8.7330\times10^{-12}$	e_0	-4.6206×10^{-13}
a_3	$+6.6228\times10^{-10}$			c_3	-5.4481×10^{-14}	e_1	$+1.8676\times10^{-14}$
						e_2	-2.1687×10^{-16}

and the value at pressure P is given by

$$\begin{aligned} K = K_0 & \\ &+ P\,(3.239908 + 1.43713\times10^{-3}t + 1.16092\times10^{-4}t^2 - 5.77905\times10^{-7}t^3) \\ &+ P\,(2.2838\times10^{-3} - 1.0981\times10^{-5}t - 1.6078\times10^{-6}t^2)\,S \\ &+ P\,(1.91075\times10^{-4})\,S^{\frac{3}{2}} \\ &+ P^2\,(8.50935\times10^{-5} - 6.12293\times10^{-6}t + 5.2787\times10^{-8}t^2) \\ &+ P^2\,(-9.9348\times10^{-7} + 2.0816\times10^{-8}t + 9.1697\times10^{-10}t^2)\,S. \end{aligned} \tag{4.6}$$

When potential temperature (θ) is available, it should be converted to *in situ* temperature. The conversion equation is obtained as follows using the adiabatic lapse rate $\Gamma(t,S,P)$:

$$t(\theta_0,S,P) = \theta_0 + \int_{P_0}^{P} \Gamma(t,S,P')dP'. \tag{4.7}$$

A polynomial for the adiabatic lapse rate $\Gamma(t,S,P)$ is given by UNESCO (1983):

$$\begin{aligned} \Gamma(t,S,P) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ &+ (b_0 + b_1 t)(S-35) \\ &+ \{c_0 + c_1 t + c_2 t^2 + c_3 t^3 + (d_0 + d_1 t)(S-35)\}\,P \\ &+ (e_0 + e_1 t + e_2 t^2)\,P^2. \end{aligned} \tag{4.8}$$

Coefficients are given on Table 4.1.

4.2 An equation of state used by MRI.COM

4.2.1 Formulation

Since potential temperature (θ) is a prognostic variable in MRI.COM, it is desirable that the potential temperature can be directly used for the equation of state in the model. If this is possible, it will not be necessary to convert between potential temperature and *in situ* temperature. Accordingly, the equation of state of MRI.COM is a polynomial of almost the same form as UNESCO, but it is now the function of θ, S, and P and has a modified set of parameters. The parameters are determined by the least square fit for a realistic range of potential temperature and salinity following the method of Ishizaki (1994).

In MRI.COM version 4, considering the treatment of brackish waters in coastal modeling as well as the practical demand of numerical stability, we slightly modified the polynomial form and revised the set of coefficients relative to the previous version. The process of determining coefficients will be described below. We allow salinity to take slightly negative values that could be caused by numerical errors. So we use a slightly wider range ($-2\le\theta\le40$ °C, $-2\le S\le42$ pss, and $0\le P\le1000$ bar) than the previous version.

First, density is calculated at the sea surface (potential density or σ_θ) using equations (4.1) and (4.2). We introduce a minor modification from $(S)^{\frac{3}{2}}$ to $(|S|)^{\frac{3}{2}}$. It has been confirmed that this modification leads to a smooth density profile at $S=0$.

The pressure dependent part, or specific volume $K(\theta,S,P)$ is given by

$$\begin{aligned} K(\theta,S,P) &= e_1(P) + e_2(P)\theta + e_3(P)\theta^2 + e_4(P)\theta^3 + e_5(P)\theta^4 \\ &+ S(f_1(P) + f_2(P)\theta + f_3(P)\theta^2 + f_4(P)\theta^3) \\ &+ (|S|)^{\frac{3}{2}}(f_5(P) + f_6(P)\theta + f_7(P)\theta^2), \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} e_1(P) &= ec_1 + (gc_1 + hc_1P)P, & f_1(P) &= fc_1 + (gc_5 + hc_4P)P, \\ e_2(P) &= ec_2 + (gc_2 + hc_2P)P, & f_2(P) &= fc_2 + (gc_6 + hc_5P)P, \\ e_3(P) &= ec_3 + (gc_3 + hc_3P)P, & f_3(P) &= fc_3 + (gc_7 + hc_6P)P, \\ e_4(P) &= ec_4 + gc_4P, & f_4(P) &= fc_4, \\ e_5(P) &= ec_5, & f_5(P) &= fc_5 + gc_8P, \\ & & f_6(P) &= fc_6, \\ & & f_7(P) &= fc_7. \end{aligned} \tag{4.10}$$

The set of coefficients in the above equation is computed using a least square fit and listed on Table 4.2. In the previous version, uniform 1.0 °C, 1.0 pss, 10 decibar bins were used for the least square fit in the range of $-2 \leq \theta \leq 40$ °C, $0 \leq S \leq 42$ pss, and $0 \leq P \leq 1000$ bar. But it is found that maximum error tends to occur in the range of low potential temperature and low salinity, and increasing the number of bins in this range is favorable for reducing the error (not shown). Therefore, potential temperature and salinity bins are changed from 1.0 °C and 1.0 pss to 0.1 °C and 0.1 pss for $-2 \leq \theta \leq 10$ °C and $-2 \leq S \leq 10$ pss, respectively. Using 151×151×101 combinations of the above range of potential temperature, salinity, and pressure, *in situ* temperature is first computed using (4.7). Density is then calculated by the UNESCO equations using *in situ* temperature and salinity. The above coefficients are determined using these data of density, potential temperature, salinity, and pressure by the least square method. They are given on Table 4.2.

Table4.2 Coeffients for the equation of state of sea water. See equation (4.10).

ec_1	19659.35	fc_1	52.85624	gc_1	3.185918	hc_1	2.111102×10^{-4}
ec_2	144.5863	fc_2	-3.128126×10^{-1}	gc_2	2.189412×10^{-2}	hc_2	-1.196438×10^{-5}
ec_3	−1.722523	fc_3	6.456036×10^{-3}	gc_3	-2.823685×10^{-4}	hc_3	1.364330×10^{-7}
ec_4	1.019238×10^{-2}	fc_4	-5.370396×10^{-5}	gc_4	1.715739×10^{-6}	hc_4	-2.048755×10^{-6}
ec_5	-4.768276×10^{-5}	fc_5	3.884013×10^{-1}	gc_5	6.703377×10^{-3}	hc_5	6.375979×10^{-8}
		fc_6	9.116446×10^{-3}	gc_6	-1.839953×10^{-4}	hc_6	5.240967×10^{-10}
		fc_7	-4.628163×10^{-4}	gc_7	1.912264×10^{-7}		
				gc_8	1.477291×10^{-4}		

With the new set of coefficients, the maximum density error (thick dashed line) relative to the UNESCO equation is less than 1.6×10^{-3} kg m^{-3} when pressure is less than 5000 dbar (Figure 4.1). This maximum error is less than that (3×10^{-3} kg m^{-3}) of McDougall et al. (2003)'s equation of state. The error is also smaller than that using the previous version (version 3 and earlier) in this pressure range (thin dashed line). Though the relatively larger maximum errors are found when the pressure is greater than 8000 dbar, they occur only when S ≃ 40 pss or S ≃ 0 pss (not shown) and would not cause a serious problem. The standard deviation (thick solid line) is less than 1×10^{-3} kg m^{-3} when pressure is less than 8000 dbar.

4.2.2 Implementation

In MRI.COM, *in situ* density is computed by (4.3) using (4.2) for ρ_{surf} and (4.9) for K. For Pressure (P), a horizontally uniform value is used on each vertical level where densities are evaluated, instead of using the actual pressure at each location. This is based on the assumption that horizontal variation of pressure on a constant depth (z^*) surface does not affect the required accuracy of density.

In the program code of MRI.COM, the *in situ* density is evaluated at the depth where the tracer is defined and the depth of the boundary of tracer cells. In the standard case, pressure at the depth where tracer is defined ($\overline{P_{k-\frac{1}{2}}}$) and that at the depth of the cell boundary ($\overline{P_k}$) are calculated as follows:

$$10^6\overline{P_{k-\frac{1}{2}}} = \rho_0 g z_{k-\frac{1}{2}}, \tag{4.11}$$

$$10^6\overline{P_k} = \rho_0 g z_k, \tag{4.12}$$

where ρ_0 is the reference density, g is the acceleration due to gravity (Table 2.1), and $z_{k-\frac{1}{2}}$ and z_k are the actual depths in the state of rest. Note that a factor 10^6 converts pressure in cgs units (dyn cm^{-2}) to bar.

If `CALPP` option is chosen, the time variation of these horizontally uniform pressures is considered. In this case, pressures are evaluated as follows:

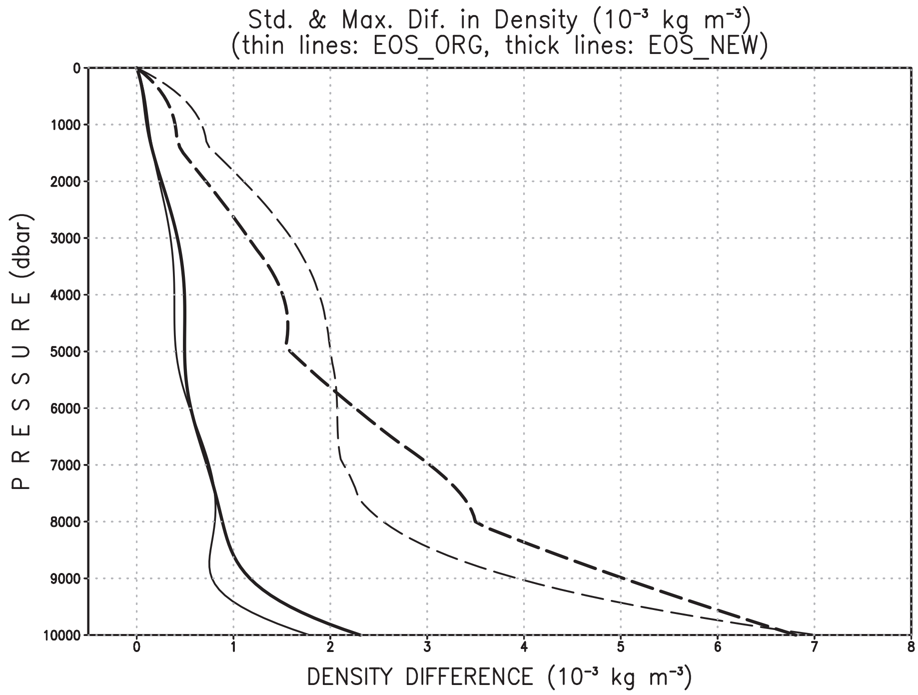

Figure4.1 Deviation from the UNESCO equation for the old (version 3 and earlier) and new (version 4) set of coefficients. Unit is $10^{-3}\,\mathrm{kg\,m^{-3}}$. Thin lines represent the equation of state using the old coefficients and thick lines represent those of the new ones. The solid (long dashed) lines represent the standard deviation (maximum error) from the UNESCO equation in the range of $-2^\circ \le \theta \le 10\,^\circ\mathrm{C}$ and $10 \le S \le 40\,\mathrm{pss}$.

$$10^6\overline{P_{\frac{1}{2}}} = \frac{1}{2} g \Delta z_{\frac{1}{2}} \overline{\rho_{\frac{1}{2}}}^{xy}, \tag{4.13}$$

$$10^6\overline{P_{k-\frac{1}{2}}} = 10^6 P_{\frac{1}{2}} + g \sum_{l=2}^{k} \Delta z_{l-1} \frac{\overline{\rho_{l-\frac{3}{2}}}^{xy} + \overline{\rho_{l-\frac{1}{2}}}^{xy}}{2} \quad (k \ge 2), \tag{4.14}$$

$$10^6\overline{P_k} = g \sum_{k=1}^{k} \Delta z_{k-\frac{1}{2}} \overline{\rho_{k-\frac{1}{2}}}^{xy}. \tag{4.15}$$

where $\overline{\rho_{k-\frac{1}{2}}}^{xy}$ represents horizontally averaged density at $(k-\frac{1}{2})$th level. In addition, the density averaged for the entire model domain ($\bar{\rho}$) is used as the reference density in the momentum equations instead of ρ_0.

These quantities are evaluated at a time step interval specified by namelist `nml_calpp` (Table 4.3) and should be stored in a restart file to be used in the next run (Table 4.4). In the restart file, horizontally averaged density at tracer levels ($\overline{\rho_{k-\frac{1}{2}}}^{xy}$), the density averaged for the entire model domain ($\bar{\rho}$), pressures used for the equation of state ($\overline{P_{k-\frac{1}{2}}}$ and $\overline{P_k}$) are stored. The restart file has the following format:

Format of the restart file for the equation of state (nmlrs_density).

```
read(8)       :: dmn(km), ddmna, pm(km+1), pd(km)
integer(4)    :: k, reck
real(8)       :: density_array(3*km+2)
character(14) :: date = '20010101000000'  !- for 0:00z1JAN2001
integer(4),parameter :: nu = 10 ! device number

open(nu, file='result/rs_density.'//date, form='unformatted', &
     & access='direct',recl=(3*km+2)*8)
read(nu, rec=1) density_array
close(nu)

reck = 0
do k = 1, km
  reck = reck + 1
  dmn(k) = density_array(reck) ! horizontally averaged density
end do
reck = reck + 1
ddmna = density_array(reck)    ! density averaged over the entire model domain
do k = 1, km + 1
  reck = reck + 1
  pm(k) = density_array(reck) ! pressure at the tracer cell boundaries
end do
do k = 1, km
  reck = reck + 1
  pd(k) = density_array(reck) ! pressure at the tracer levels
end do
```

Table4.3 namelist nml_calpp

variable name	units	description	usage
nstep_calpp_interval	1	time step interval with which horizontally averaged density and pressure are evaluated for the equation of state	required if CALPP

Table4.4 namelist nml_density_run

variable name	units	description	usage
l_rst_density_in	logical	.true. : Read restart files specified by nmlrs_density for the initial condition. .false. : Start condition is calculated by the 3-D density field of the initial state.	default = l_rst_in of nml_run_ini_state, see Table 21.5.

Chapter 5

Continuity equation

5.1 Introduction

The mass (volume) fluxes, which are fundamental for estimating the advection of momentum and tracers, are calculated on the basis of the finite difference expression of the continuity equation. Owing to the use of staggered grid arrangement, the finite difference expression of the continuity equation (2.88) differs for the T-cell and U-cell (Figure 5.1). In MRI.COM, the mass continuity for the T-cell is primary and that for the U-cell is derived from the former by an averaging operation. By this, we can avoid spurious vertical mass fluxes for the U-cell continuity, which appear when the U-cell continuity is calculated independently of the T-cell continuity, with the largest error magnitude increasing as the grid size decreases (Webb, 1995).

5.2 Finite difference expression for the T-cell continuity equation

Finite difference expressions of the free surface equation (2.40),

$$\frac{\partial(\eta+H)}{\partial t}+\frac{1}{h_\mu h_\psi}\left[\frac{\partial(h_\psi U)}{\partial\mu}+\frac{\partial(h_\mu V)}{\partial\psi}\right]=P-E+R+I, \tag{5.1}$$

the continuity equation (2.36),

$$\frac{\partial z_s}{\partial t}+\frac{1}{h_\mu h_\psi}\left\{\frac{\partial(z_s h_\psi u)}{\partial\mu}+\frac{\partial(z_s h_\mu v)}{\partial\psi}\right\}+\frac{\partial(z_s\dot{s})}{\partial s}=0, \tag{5.2}$$

and the tracer equations (2.37, 2.38)

$$\frac{\partial(z_s\theta)}{\partial t}+\frac{1}{h_\mu h_\psi}\left\{\frac{\partial(z_s h_\psi u\theta)}{\partial\mu}+\frac{\partial(z_s h_\mu v\theta)}{\partial\psi}\right\}+\frac{\partial(z_s\dot{s}\theta)}{\partial s}=-z_s\nabla\cdot\mathrm{F}_\theta+z_s Q_\theta, \tag{5.3}$$

$$\frac{\partial(z_s S)}{\partial t}+\frac{1}{h_\mu h_\psi}\left\{\frac{\partial(z_s h_\psi uS)}{\partial\mu}+\frac{\partial(z_s h_\mu vS)}{\partial\psi}\right\}+\frac{\partial(z_s\dot{s}S)}{\partial s}=-z_s\nabla\cdot\mathrm{F}_S+z_s Q_S, \tag{5.4}$$

must be mutually consistent in order to keep sign-definiteness of tracers. Among them, the continuity equation is fundamental. The continuity equation is used to diagnostically obtain vertical velocity $\dot{s}$. Actually, in MRI.COM, vertical velocity is not computed but vertical transport (`wlwl`) is extensively used. Vertical transport is mathematically expressed as $W^T \equiv z_s\dot{s}\Delta A^T(s)$, where $\Delta A^T(s)$ is the horizontal area of a T-cell*. To obtain W^T, we horizontally integrate the semi discrete expression for the continuity equation (2.88) using the surface boundary condition for w^*.

Using the surface boundary condition for the vertical velocity in z^* coordinate (2.49), we have

$$W^T_{z^*=0}=z_s\dot{s}_{s=0}\Delta A^T_{s=0}=\frac{H+\eta}{H}w^*_{z^*=0}\Delta A^T_{z^*=0}=-(P-E+R+I)\Delta A^T_{z^*=0}. \tag{5.5}$$

The surface flux of a state variable caused by the surface fresh water transport (r.h.s. of the above) is treated as the surface boundary condition for the advection term of that state variable.

We now integrate the semi-discrete expression for the continuity equation (2.88) over the oceanic part of the vertically k-th T-cell. The integral over a T-cell of the tendency term of the scaling factor (2nd term on r.h.s.) is expressed as a

* Horizontal cross section of a T-cell is treated as constant throughout the cell, that is, vertically bounding walls are assumed to move with the cell

multiple of the average cross sectional area ($\Delta A^T_{i,j,k-\frac{1}{2}}$) and the average thickness for the k-th vertical cell in z^* coordinate ($\Delta s_{i,j,k-\frac{1}{2}}$):

$$
\begin{aligned}
W^T_{i,j,k} &= W^T_{i,j,k-1} + \left[\Delta A^T_{i,j,k-\frac{1}{2}} \Delta s_{i,j,k-\frac{1}{2}} (\partial_t z_s)_{i,j} \right] \\
&+ \frac{(\mathtt{dzu}\cdot u)_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} + (\mathtt{dzu}\cdot u)_{i+\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}}}{2} \Delta y_{i+\frac{1}{2},j} - \frac{(\mathtt{dzu}\cdot u)_{i-\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} + (\mathtt{dzu}\cdot u)_{i-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}}}{2} \Delta y_{i-\frac{1}{2},j} \\
&+ \frac{(\mathtt{dzu}\cdot v)_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} + (\mathtt{dzu}\cdot v)_{i-\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}}{2} \Delta x_{i,j+\frac{1}{2}} - \frac{(\mathtt{dzu}\cdot v)_{i+\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}} + (\mathtt{dzu}\cdot v)_{i-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}}}{2} \Delta x_{i,j-\frac{1}{2}}, \quad (5.6)
\end{aligned}
$$

where it is assumed that the actual thickness of a U-cell (dzu) also works as a land-sea mask. The second term on the right hand side is regarded as an actual temporal variation of the volume of a T-cell, that is,

$$
\Delta A^T_{i,j,k-\frac{1}{2}} \Delta s_{i,j,k-\frac{1}{2}} (\partial_t z_s)_{i,j} = \partial_t (\mathtt{volt})_{i,j,k-\frac{1}{2}}, \quad (5.7)
$$

where equation (3.28) is used. Conversely, by using W^T calculated from (5.6), calculation of the volume flux divergence for a T-cell results in the volume change rate of the T-cell.

To check consistency with the free surface equation, we insert (5.7) into (5.6) and then sum them up from the surface to the bottom to have

$$
\begin{aligned}
W^T_{i,j,ktbtm} = 0 = &- (P - E + R + I) \Delta A^T_{i,j,\frac{1}{2}} + \frac{\partial (V^T_{\mathrm{column}})_{i,j}}{\partial t} + \frac{U_{i+\frac{1}{2},j+\frac{1}{2}} + U_{i+\frac{1}{2},j-\frac{1}{2}}}{2} \Delta y_{i+\frac{1}{2},j} \\
&- \frac{U_{i-\frac{1}{2},j+\frac{1}{2}} + U_{i-\frac{1}{2},j-\frac{1}{2}}}{2} \Delta y_{i-\frac{1}{2},j} + \frac{V_{i+\frac{1}{2},j+\frac{1}{2}} + V_{i-\frac{1}{2},j+\frac{1}{2}}}{2} \Delta x_{i,j+\frac{1}{2}} - \frac{V_{i+\frac{1}{2},j-\frac{1}{2}} + V_{i-\frac{1}{2},j-\frac{1}{2}}}{2} \Delta x_{i,j-\frac{1}{2}}. \quad (5.8)
\end{aligned}
$$

Because the tendency of the volume of a T-point water column V^T_{column} is determined by the undulation of sea level, we have,

$$
\frac{\partial (V^T_{\mathrm{column}})_{i,j}}{\partial t} = (\mathtt{areat})_{i,j,\frac{1}{2}} \frac{\partial \eta_{i,j}}{\partial t} = (\mathtt{ws})_{i,j} + (\mathtt{transport\_wflux})_{i,j}, \quad (5.9)
$$

where

$$
\begin{aligned}
(\mathtt{ws})_{i,j} = -\Bigg(&\frac{U_{i+\frac{1}{2},j+\frac{1}{2}} + U_{i+\frac{1}{2},j-\frac{1}{2}}}{2} \Delta y_{i+\frac{1}{2},j} - \frac{U_{i-\frac{1}{2},j+\frac{1}{2}} + U_{i-\frac{1}{2},j-\frac{1}{2}}}{2} \Delta y_{i-\frac{1}{2},j} \\
&+ \frac{V_{i+\frac{1}{2},j+\frac{1}{2}} + V_{i-\frac{1}{2},j+\frac{1}{2}}}{2} \Delta x_{i,j+\frac{1}{2}} - \frac{V_{i+\frac{1}{2},j-\frac{1}{2}} + V_{i-\frac{1}{2},j-\frac{1}{2}}}{2} \Delta x_{i,j-\frac{1}{2}} \Bigg) \quad (5.10)
\end{aligned}
$$

and

$$
(\mathtt{transport\_wflux})_{i,j} = (P - E + R + I)_{i,j} \Delta A^T_{i,j,\frac{1}{2}}. \quad (5.11)
$$

It is confirmed that (5.8) has an integrated form of the free surface equation (2.40).

As a preparation for explaining the U-cell continuity equation, the finite difference expression of the continuity equation for the vertically k-th T-cell (5.6) is rewritten in a concise form as follows, by defining the mass fluxes passing through each side of the grid cell (Figure 5.1):

$$
\begin{aligned}
MC^T_{i,j,k-\frac{1}{2}} &\equiv U^T_{i-\frac{1}{2},j} - U^T_{i+\frac{1}{2},j} + V^T_{i,j-\frac{1}{2}} - V^T_{i,j+\frac{1}{2}} + W^T_{i,j,k} - W^T_{i,j,k-1} \\
&= \partial_t (\mathtt{volt})_{i,j,k-\frac{1}{2}}, \quad (5.12)
\end{aligned}
$$

where

$$
U^T_{i+\frac{1}{2},j} = u^*_{i+\frac{1}{2},j} \Delta y_{i+\frac{1}{2},j} \Delta z, \quad V^T_{i,j+\frac{1}{2}} = v^*_{i,j+\frac{1}{2}} \Delta x_{i,j+\frac{1}{2}} \Delta z, \quad (5.13)
$$

$$
u^*_{i+\frac{1}{2},j} = \frac{1}{2}(u_{i+\frac{1}{2},j+\frac{1}{2}} + u_{i+\frac{1}{2},j-\frac{1}{2}}), \quad v^*_{i,j+\frac{1}{2}} = \frac{1}{2}(v_{i+\frac{1}{2},j+\frac{1}{2}} + v_{i-\frac{1}{2},j+\frac{1}{2}}). \quad (5.14)
$$

The finite difference analog of the continuity for the partial T-cell along the coastline (Figure 5.1b) is defined as follows:

$$
V^T_{i,j+\frac{1}{2}} = \frac{1}{2} v^*_{i,j+\frac{1}{2}} \Delta x_{i,j+\frac{1}{2}} \Delta z, \quad V^T_{i,j-\frac{1}{2}} = \frac{1}{2} v^*_{i,j-\frac{1}{2}} \Delta x_{i,j-\frac{1}{2}} \Delta z, \quad (5.15)
$$

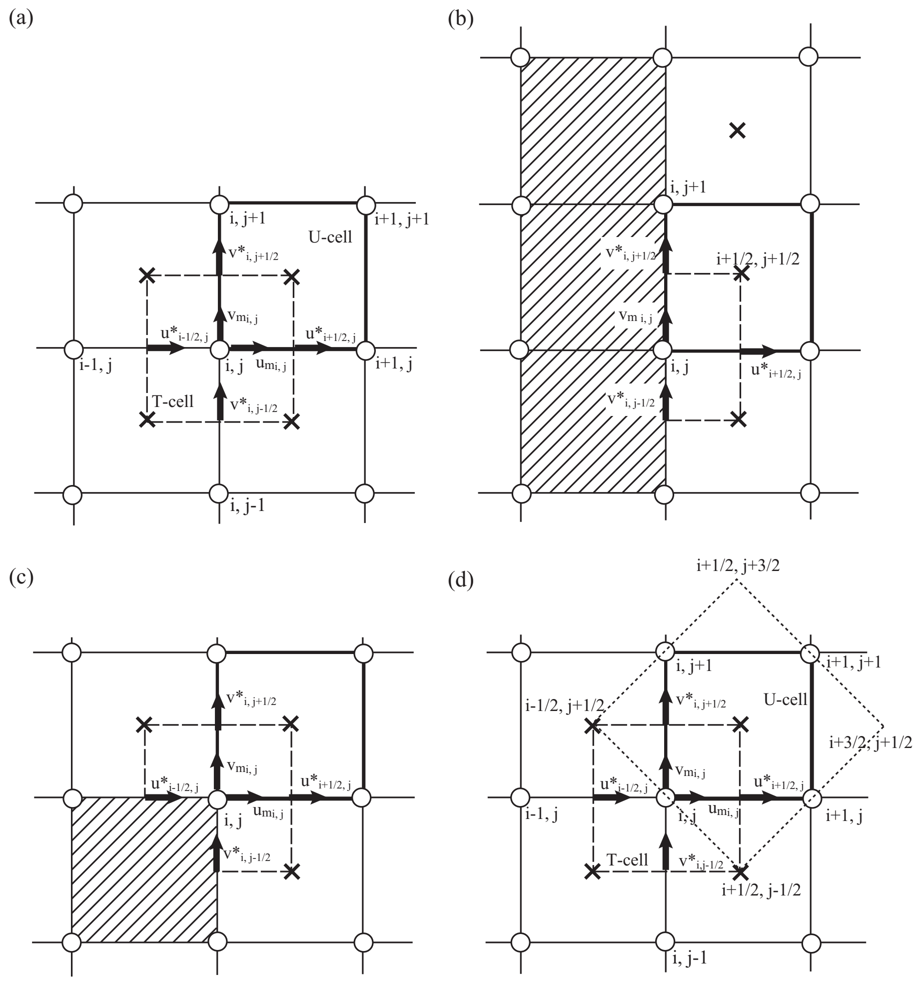

Figure5.1 Horizontal arrangement of variables for the continuity equation. (a) Relationship between T-cell and U-cell (standard form). (b),(c) Relationship between T-cell and U-cell near the coast. (d) Diagonal square grid cell and mass fluxes.

$$v^*_{i,j+\frac{1}{2}} = v_{i+\frac{1}{2},j+\frac{1}{2}}, \qquad v^*_{i,j-\frac{1}{2}} = v_{i+\frac{1}{2},j-\frac{1}{2}}, \qquad u^*_{i-\frac{1}{2},j} = 0. \tag{5.16}$$

For the corner part of land as shown in Figure 5.1(c), it is given as follows:

$$U^T_{i-\frac{1}{2},j} = \frac{1}{2}u^*_{i-\frac{1}{2},j}\Delta y_{i-\frac{1}{2},j}\Delta z, \quad V^T_{i,j-\frac{1}{2}} = \frac{1}{2}v^*_{i,j-\frac{1}{2}}\Delta x_{i,j-\frac{1}{2}}\Delta z, \tag{5.17}$$

$$u^*_{i-\frac{1}{2},j} = u_{i-\frac{1}{2},j+\frac{1}{2}}, \quad v^*_{i,j-\frac{1}{2}} = v_{i+\frac{1}{2},j-\frac{1}{2}}. \tag{5.18}$$

The boundary condition for $W^T_{i,j}$ is as follows:

$$W^T_{i,j,0} = (\Delta A^T \partial_t \eta)_{i,j} = \left(U_{i-\frac{1}{2},j}\Delta y_{i-\frac{1}{2},j} - U_{i+\frac{1}{2},j}\Delta y_{i+\frac{1}{2},j} + V_{i,j-\frac{1}{2}}\Delta x_{i,j-\frac{1}{2}} - V_{i,j+\frac{1}{2}}\Delta x_{i,j+\frac{1}{2}}\right) - (P-E+R+I)_{i,j}\,\Delta A^T_{i,j,\frac{1}{2}} \tag{5.19}$$

at the surface (where $U = \sum_{k=1}^{N} u_{k-\frac{1}{2}}\Delta z_{k-\frac{1}{2}}$ and $V = \sum_{k=1}^{N} v_{k-\frac{1}{2}}\Delta z_{k-\frac{1}{2}}$), and

$$W^T_{i,j,ktbtm} = 0 \tag{5.20}$$

at the bottom ($k = ktbtm$).

5.3 Finite difference expression for the U-cell continuity equation

The finite difference expression of the continuity equation for a U-cell $(i+\frac{1}{2}, j+\frac{1}{2})$ is defined using those for T-cells as follows (Figure 5.1(a, b, and c)):

$$\begin{aligned} MC^U_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} &\equiv \frac{MC^T_{i,j,k-\frac{1}{2}}}{N_{i,j,k-\frac{1}{2}}} + \frac{MC^T_{i+1,j,k-\frac{1}{2}}}{N_{i+1,j,k-\frac{1}{2}}} + \frac{MC^T_{i,j+1,k-\frac{1}{2}}}{N_{i,j+1,k-\frac{1}{2}}} + \frac{MC^T_{i+1,j+1,k-\frac{1}{2}}}{N_{i+1,j+1,k-\frac{1}{2}}} \\ &= \frac{\partial_t(\Delta V^T)_{i,j,k-\frac{1}{2}}}{N_{i,j,k-\frac{1}{2}}} + \frac{\partial_t(\Delta V^T)_{i+1,j,k-\frac{1}{2}}}{N_{i+1,j,k-\frac{1}{2}}} + \frac{\partial_t(\Delta V^T)_{i,j+1,k-\frac{1}{2}}}{N_{i,j+1,k-\frac{1}{2}}} + \frac{\partial_t(\Delta V^T)_{i+1,j+1,k-\frac{1}{2}}}{N_{i+1,j+1,k-\frac{1}{2}}}, \end{aligned} \tag{5.21}$$

where $N_{i,j,k-\frac{1}{2}}$ is the number of sea grid cells around the T-point (i, j) in the $(k-\frac{1}{2})$th layer. Usually, $N = 4$ for T-cells away from land (Figure 5.1a), but $N < 4$ for the partial T-cells along coast lines (Figure 5.1b,c). This equation means that the mass convergence in a U-cell consists of the sum of the contributions from four surrounding T-cells.

The standard form of the mass continuity, which applies for U-cells $(i+\frac{1}{2}, j+\frac{1}{2})$ away from coast lines, is as follows:

$$\begin{aligned} MC^U_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} &\equiv \frac{1}{4}(MC^T_{i,j,k-\frac{1}{2}} + MC^T_{i+1,j,k-\frac{1}{2}} + MC^T_{i,j+1,k-\frac{1}{2}} + MC^T_{i+1,j+1,k-\frac{1}{2}}) \\ &= \frac{1}{4}[\partial_t(\Delta V^T)_{i,j,k-\frac{1}{2}} + \partial_t(\Delta V^T)_{i+1,j,k-\frac{1}{2}} + \partial_t(\Delta V^T)_{i,j+1,k-\frac{1}{2}} + \partial_t(\Delta V^T)_{i+1,j+1,k-\frac{1}{2}}]. \end{aligned} \tag{5.22}$$

This can be rewritten as

$$\begin{aligned} &\frac{1}{2}(U^U_{i,j} + U^U_{i,j+1}) - \frac{1}{2}(U^U_{i+1,j} + U^U_{i+1,j+1}) + \frac{1}{2}(V^U_{i,j} + V^U_{i+1,j}) - \frac{1}{2}(V^U_{i,j+1} + V^U_{i+1,j+1}) \\ &\qquad - W^U_{i+\frac{1}{2},j+\frac{1}{2},k-1} + W^U_{i+\frac{1}{2},j+\frac{1}{2},k} \simeq \partial_t(\Delta V^U)_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}. \end{aligned} \tag{5.23}$$

Note that the l.h.s. does not give the exact volume tendency that would be obtained by following the definition of the volume of a U-cell (3.26). This results in a local violation of momentum conservation, though the momentum is globally conserved. We keep using the definition of the volume of a U-cell given by (3.26) because we do not want to use the simple average of the volumes of T-cells similar to (5.21) for defining the control volume of a U-cell. This comes from the need for a simple algebraic expression. Terms $U^U_{i,j}$, $V^U_{i,j}$, and $W^U_{i+\frac{1}{2},j+\frac{1}{2},k}$ are defined as follows:

$$U^U_{i,j} = u_{mi,j}\Delta y_{i,j}\Delta z, \quad V^U_{i,j} = v_{mi,j}\Delta x_{i,j}\Delta z, \tag{5.24}$$

$$u_{mi,j} = \frac{1}{2}(u^*_{i+\frac{1}{2},j} + u^*_{i-\frac{1}{2},j}), \quad v_{mi,j} = \frac{1}{2}(v^*_{i,j+\frac{1}{2}} + v^*_{i,j-\frac{1}{2}}), \tag{5.25}$$

$$W^U_{i+\frac{1}{2},j+\frac{1}{2},k} = \frac{1}{4}(W^T_{i,j,k} + W^T_{i+1,j,k} + W^T_{i,j+1,k} + W^T_{i+1,j+1,k}). \tag{5.26}$$

Note that W^U is obtained by an averaging operation on W^T and also that the following equations are derived from (5.13), (5.24), and (5.25) for the standard form of the U-cell continuity:

$$\begin{aligned} U^U_{i,j} &= \frac{1}{2}(U^T_{i+\frac{1}{2},j} + U^T_{i-\frac{1}{2},j}), \\ V^U_{i,j} &= \frac{1}{2}(V^T_{i,j+\frac{1}{2}} + V^T_{i,j-\frac{1}{2}}). \end{aligned} \tag{5.27}$$

The advecting velocity for momentum is based on (U^U, V^U, W^U).

All the above relationships hold for the cases with variable grid sizes (Figure 3.1c).

The l.h.s. of the standard form of the continuity equation (5.23) expresses the convergence of mass fluxes along the horizontal coordinate axes and it is completed as far as the continuity equation is concerned. However, when the mass continuity is used to calculate the momentum advection, the l.h.s. of (5.23) is rewritten as follows to express the convergence of the diagonal mass fluxes to the coordinate axes, and is used together with its original form (5.23):

$$\begin{aligned} &\frac{1}{2}(U^U_{i,j} + V^U_{i,j}) - \frac{1}{2}(U^U_{i+1,j+1} + V^U_{i+1,j+1}) + \frac{1}{2}(U^U_{i,j+1} - V^U_{i,j+1}) - \frac{1}{2}(U^U_{i+1,j} - V^U_{i+1,j}) \\ &\qquad - W^U_{i+\frac{1}{2},j+\frac{1}{2},k-1} + W^U_{i+\frac{1}{2},j+\frac{1}{2},k} \simeq \partial_t(\Delta V^U)_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}} \end{aligned} \tag{5.28}$$

Let us explain the meaning taking the first term on the l.h.s. of (5.28),

$$U^U_{i,j} + V^U_{i,j} = \frac{1}{2}\{(U^T_{i-\frac{1}{2},j} + V^T_{i,j-\frac{1}{2}}) + (U^T_{i+\frac{1}{2},j} + V^T_{i,j+\frac{1}{2}})\}, \tag{5.29}$$

as an example, where (5.27) is used. If the flow is horizontally nondivergent, the horizontal mass fluxes $U^T_{i-\frac{1}{2},j}$ and $V^T_{i,j-\frac{1}{2}}$ in the first term on the r.h.s. are expressed by streamfunction at two pairs of U-points, $(i-\frac{1}{2}, j+\frac{1}{2})$ and $(i-\frac{1}{2}, j-\frac{1}{2})$, and $(i-\frac{1}{2}, j-\frac{1}{2})$ and $(i+\frac{1}{2}, j-\frac{1}{2})$, respectively. Then, their sum corresponds to the net mass flux crossing the diagonal section connecting the two U-points $(i-\frac{1}{2}, j+\frac{1}{2})$ and $(i+\frac{1}{2}, j-\frac{1}{2})$ (Figure 5.1d). The second term on the r.h.s. expresses the same quantity, though the route is different. Thus, multiplying by a factor of two, the l.h.s. of (5.28) means the horizontal mass convergence in the diagonal square defined by four U-points $(i-\frac{1}{2}, j+\frac{1}{2})$, $(i+\frac{1}{2}, j-\frac{1}{2})$, $(i+\frac{3}{2}, j+\frac{1}{2})$, and $(i+\frac{1}{2}, j+\frac{3}{2})$. Multiplying by a factor of $\frac{1}{2}$, the l.h.s. of (5.28) itself means the horizontal mass convergence in the U-cell $(i+\frac{1}{2}, j+\frac{1}{2})$, whose area is a half of that of the diagonal square.

Taking the first four terms on the l.h.s. of (5.23) as $A_{i+\frac{1}{2},j+\frac{1}{2}}$ and those of (5.28) as $B_{i+\frac{1}{2},j+\frac{1}{2}}$, the standard form of the continuity equation for the U-cell used for the calculation of the momentum advection is generally expressed as:

$$\alpha\ A_{i+\frac{1}{2},j+\frac{1}{2}} + \beta\ B_{i+\frac{1}{2},j+\frac{1}{2}} - W^U_{i+\frac{1}{2},j+\frac{1}{2},k-1} + W^U_{i+\frac{1}{2},j+\frac{1}{2},k} \simeq \partial_t(\Delta V^U)_{i+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2}}, \tag{5.30}$$

where

$$\alpha + \beta = 1. \tag{5.31}$$

As shown later in Chapter 7, $(\alpha, \beta) = (2/3, 1/3)$ for the generalized Arakawa scheme and $(\alpha, \beta) = (1/2, 1/2)$ for the standardized form derived from the continuity equation generalized for arbitrary bottom topography. What Webb (1995) proposed corresponds to $(\alpha, \beta) = (1, 0)$.

5.4 Summary

To summarize the solution procedure, MRI.COM uses (5.8) to obtain W^T under the boundary conditions (5.19) and (5.20). Then, using W^T, W^U is obtained by (5.26).