D. Finite discretization form and pressure equation solver

D-1. Grid structure

The model grid structure is the Arakawa-C type in the horizontal direction and the Lorenz-type in the vertical direction, which is the same as in Ikawa and Saito (1991). This chapter briefly describes the structure of the staggered grid, supplementing Ikawa and Saito (1991)

D-1-1 Structure of the staggered grid

All variables, other than velocity components, advection terms and metric tensors, are defined at the "scalar" grid point indexed by integer (i, j, k). Velocity components U, V and W (as well as W^*) are located at the grid points indexed by the half integer $(i+1/2, j, k)$, $(i, j+1/2, k)$, $(i, j, k+1/2)$. Advection terms $ADVU$, $ADVV$ and $ADVW$ are computed at the same points as U, V and W, respectively. Metric tensor $G^{1/2}$ is defined at the grid point (i, j) independently of k, while $G^{1/2}G^{13}$ and $G^{1/2}G^{23}$ are computed at $(i+1/2, j, k+1/2)$, $(i, j+1/2, k+1/2)$, respectively (Fig. D1-1-1).

Figures D1-1-2 and D1-1-3 present horizontal and vertical cross sections of the grid mesh of the model. The physical boundaries are located at $x=1+1/2$ and $x=nx-1/2$ in the x-direction, where the x-component of velocity is $U(2,,)$ and $U(NX,,)$. It is also the case in the y-direction, and the y-component of velocity is $V(,2,)$ and $V(,NY,)$ at $y=1+1/2$ and $y=ny-1/2$. However, in the vertical direction, the physical upper and lower boundaries are located at $z=1+1/2$ and $z=nz-1/2$, where the z-components of velocity are $W(,,1)$ and $W(,,NZ-1)$.

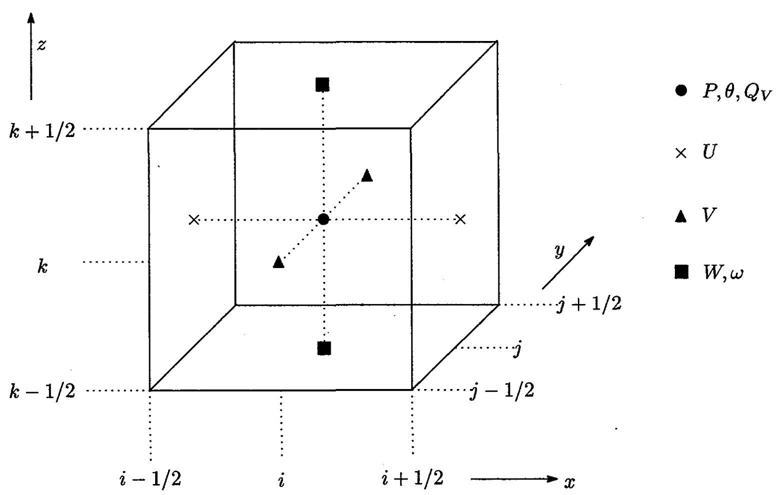

Fig. D1-1-1 Staggered grid. Reproduced from Ikawa and Saito (1991).

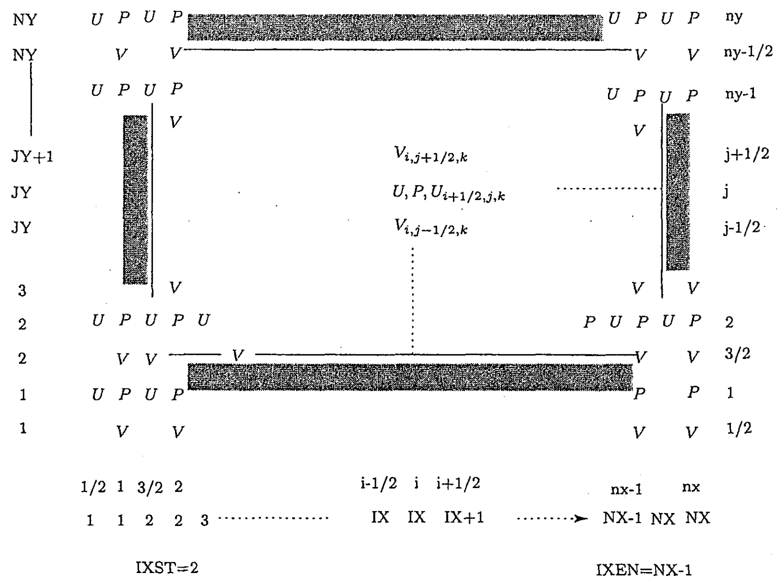

Fig. D1-1-2 Horizontal cross section of the grid mesh and the domain boundary. Reproduced from Ikawa and Saito (1991).

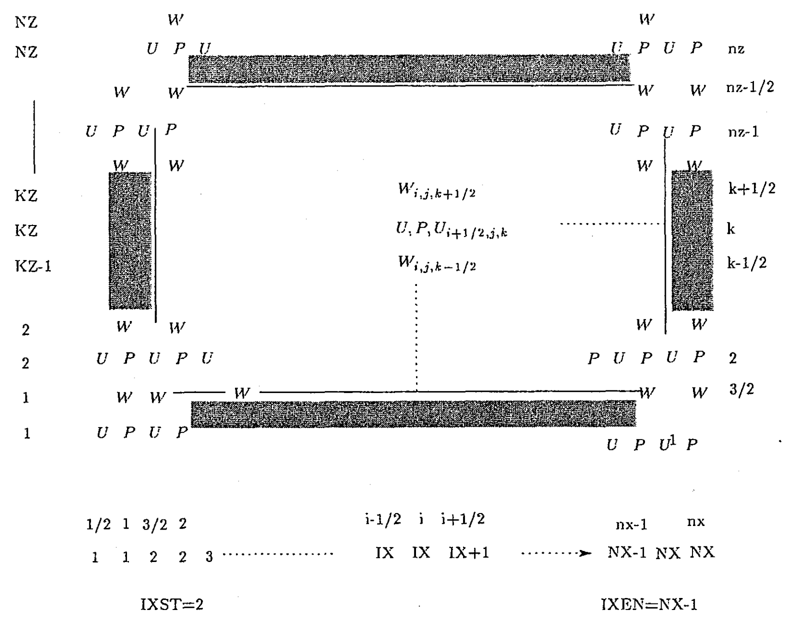

Fig. D1-1-3 Vertical cross section of the grid mesh and the domain boundary. Reproduced from Ikawa and Saito (1991).

D-1-2 Variable vertical grid

Figure D1-2-1 shows the variable grid structure in the z-direction. Two kinds of grid intervals are defined. Δz_k represents the grid intervals between the two grid points (i, j, k-1/2) and (i, j, k+1/2). In the program, this grid interval is denoted by *VDZ(K)*, which corresponds to the interval between the levels of $W(,,K)$ and $W(,,K+1)$. $\Delta z_{k+1/2}$ represents the grid intervals between the two grid points (i, j, k) and (i, j, k+1). In the program, this grid interval is denoted by *VDZ2(K)*, which corresponds to the interval between the levels of $P(,,K)$ and $P(,,K+1)$. As shown in Fig. D1-2-1, the half level is located at the center of the full level, and Δz_k and $\Delta z_{k+1/2}$ are related

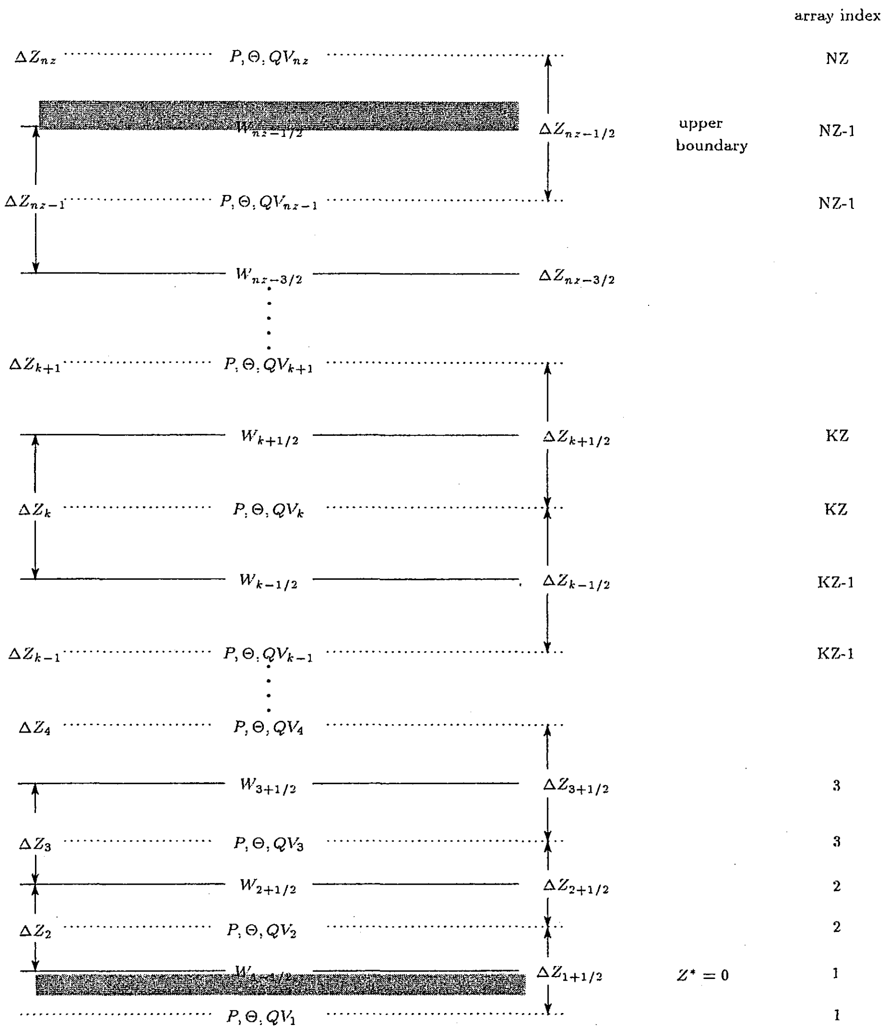

Fig. D1-2-1 Variable grid structure in the z-direction. Reproduced from Ikawa and Saito (1991).

as follows:

$$\Delta z_k = \frac{\Delta z_{k-1/2} + \Delta z_{k+1/2}}{2}, \tag{D1-2-1}$$

that is,

$$VDZ(K) = 0.5 * \{ VDZ2(K-1) + VDZ2(K) \}. \tag{D1-2-2}$$

The heights of levels k and $k+1/2$ are denoted by *ZRP(KZ)* and *ZRW(KZ)*,. The relations between the variables are given by

$$Z_{3/2} = ZRW(1) = 0, \tag{D1-2-3}$$

$$\begin{aligned} Z_{k+1/2} &= Z_{k-1/2} + \Delta z_k \\ &= ZRW(K) = \sum_{KZ=2}^{k} VDZ(KZ), \end{aligned} \tag{D1-2-4}$$

and

$$Z_1 = ZRP(1) = -\frac{\Delta z_{3/2}}{2} = -0.5 * VDZ2(1), \tag{D1-2-5}$$

$$Z_k = Z_{k-1} + \Delta z_{k-1/2}$$

$$= ZRP(K) = -0.5 * VDZ2(1) + \sum_{KZ=1}^{K-1} VDZ2(KZ). \qquad \text{(D1-2-6)}$$

Note that physical intervals between grid points (i, j, k) must be computed by multiplying $G^{1/2}(i, j)$ for both Δz_k and $\Delta z^{k+1/2}$ when the orography exists.

Program Guide (hereafter, abbreviated as P.G.)

The variable grid distances are set in sub.VRGDIS where "sub" denotes subroutine. The arrays *ZRP* and *ZRW* are defined in sub.SETZRP and sub.SETZRW.

The variable grid structure in the x- and y-directions is shown in Fig. D1-2-2. The structure is similar to that in the z-direction, but the array indexes for U and V are different from that for W in the z-direction corresponding to the locations of the lateral boundaries. For example, in the x-direction, the relations between the variables are given by

$$\Delta x_i = \frac{\Delta x_{i-1/2} + \Delta x_{i+1/2}}{2},$$

$$= VDX(I) = 0.5 * \{ VDX2(I) + VDX2(I+1) \}, \qquad \text{(D1-2-7)}$$

$$X_{3/2} = 0, \qquad \text{(D1-2-8)}$$

$$X_{i+1/2} = X_{i-1/2} + \Delta x_i$$

$$= XRU(I+1) = \sum_{IX=2}^{I} VDX(IX), \qquad \text{(D1-2-9)}$$

$$X_1 = -0.5 * VDX2(2), \qquad \text{(D1-2-10)}$$

$$X_i = X_{i-1} + \Delta x_{i-1/2}$$

$$= XRP(I) = -0.5 * VDX2(2) + \sum_{IX=2}^{I} VDX2(IX). \qquad \text{(D1-2-11)}$$

The arrays *XRP* and *XRU* are defined in relevant utilities such as initial file setting and plot, but are not currently set in the model computation. The relations in the y-direction are the same as in the x-direction.

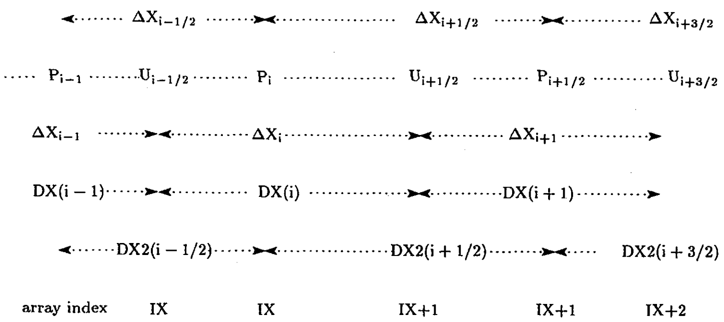

Fig. D1-2-2 Variable grid structure in x-direction. Reproduced from Ikawa and Saito (1991).

D-2. Finite discretization form

D-2-1 Finite discretization form for basic equations

Following Ikawa and Saito (1991), the averaging operator in the x-direction is defined for any variable F defined at the scalar point by

$$\overline{F}^{x}_{i+1/2}=\frac{F_i+F_{i+1}}{2}, \tag{D2-1-1}$$

and another averaging operator for any variable U defined at the grid point of a half integer by

$$\overline{U}^{x}{}_{i}=\frac{\Delta x_{i+1/2}U_{i-1/2}+\Delta x_{i-1/2}U_{i+1/2}}{\Delta x_{i-1/2}+\Delta x_{i+1/2}}$$

$$=\frac{\Delta x_{i+1/2}U_{i-1/2}+\Delta x_{i-1/2}U_{i+1/2}}{2\Delta x_i}. \tag{D2-1-2}$$

Averaging operators in the y- and z-directions are defined in the same way.

Finite differencing operators are defined by

$$\partial_x F_{i-1/2}=\frac{F_i-F_{i-1}}{\Delta x_{i-1/2}}, \tag{D2-1-3}$$

$$\partial_{2x} F_i=\frac{F_{i+1}-F_{i-1}}{2\Delta x_i}, \tag{D2-1-4}$$

$$\partial_x U_i=\frac{U_{i+1/2}-U_{i-1/2}}{\Delta x_i}. \tag{D2-1-5}$$

Using these operators, the governing equations in chapter C2 are expressed in the finite discretization form. For (C2-1-3) and (C2-1-4):

$$G^{\frac{1}{2}}G^{13}{}_{i+1/2,j,k+1/2}=(\frac{z^*{}_{k+1/2}}{H}-1)\,\partial_x z_{s_{i+1/2,j}}, \tag{D2-1-6}$$

$$G^{\frac{1}{2}}G^{23}{}_{i,j+1/2,k+1/2}=(\frac{z^*{}_{k+1/2}}{H}-1)\,\partial_y z_{s_{i,j+1/2}}. \tag{D2-1-7}$$

For (C2-1-9) - (C2-1-11):

$$U_{i+1/2,j,k}=\frac{\overline{(\rho G^{\frac{1}{2}})}^{x}}{\overline{m}^{x}}u, \tag{D2-1-8}$$

$$V_{i,j+1/2,k}=\frac{\overline{(\rho G^{\frac{1}{2}})}^{y}}{\overline{m}^{y}}v, \tag{D2-1-9}$$

$$W_{i,j,k+1/2}=\frac{\overline{(\rho G^{\frac{1}{2}})}^{z}}{m}w. \tag{D2-1-10}$$

For (C2-1-12) and (C2-1-13):

$$DIVT\,(U,V,W)_{i,j,k}=m^2(\partial_x U+\partial_y V)+m\partial_z W^*, \tag{D2-1-11}$$

$$W^*{}_{i,j,k+1/2}=\frac{1}{G^{\frac{1}{2}}}\{\,W+m\,(\overline{G^{\frac{1}{2}}G^{13}\overline{U}^{z}}^{x}+\overline{G^{\frac{1}{2}}G^{23}\overline{V}^{z}}^{y})\,\}. \tag{D2-1-12}$$

For (C2-1-15) - (C2-1-17):

$$(\frac{\partial U}{\partial t})_{i+1/2,j,k}+\partial_x P+\partial_z\{\,G^{\frac{1}{2}}G^{13}\,\overline{(\overline{P}^{z}/G^{\frac{1}{2}})}^{x}\,\}=-ADVU_{i+1/2,j,k}+RU_{i+1/2,j,k}, \tag{D2-1-13}$$

$$(\frac{\partial V}{\partial t})_{i,j+1/2,k}+\partial_y P+\partial_z\{\,G^{\frac{1}{2}}G^{23}\,\overline{(\overline{P}^{z}/G^{\frac{1}{2}})}^{y}\,\}=-ADVV_{i,j+1/2,k}+RV_{i,j+1/2,k}, \tag{D2-1-14}$$

$$\left(\frac{\partial W}{\partial t}\right)_{i,j,k+1/2}+\frac{1}{mG^{\frac{1}{2}}}\partial_z P=\frac{1}{m}\overline{BUOY}^{z}-ADVW_{i,j,k+1/2}+RW_{i,j,k+1/2}. \tag{D2-1-15}$$

Here,

$$ADVU_{i+1/2,j,k}=\overline{m}^{x}\{\partial_x(m\overline{U}^{x}\overline{u'}^{x})+\partial_y(m\overline{V}^{x}\overline{u'}^{y})\}+\partial_z(m\overline{W^*}^{x}\overline{u'}^{z})-\frac{u}{\overline{m}^{x}}\overline{PRC}^{x}, \tag{D2-1-16}$$

$$ADVV_{i,j+1/2,k}=\overline{m}^{y}\{\partial_x(m\overline{U}^{y}\overline{v'}^{x})+\partial_y(m\overline{V}^{y}\overline{v'}^{y})\}+\partial_z(m\overline{W^*}^{y}\overline{v'}^{z})-\frac{v}{\overline{m}^{y}}\overline{PRC}^{y}, \tag{D2-1-17}$$

$$ADVW_{i,j,k+1/2}=m\{\partial_x(m\overline{U}^{x}\overline{w'}^{x})+\partial_y(m\overline{V}^{z}\overline{w'}^{y})\}+\partial_z(m\overline{W^*}^{z}\overline{w'}^{z})-\frac{w}{m}PRC, \tag{D2-1-18}$$

$$RU_{i+1/2,j,k}=\overline{f_3}^{x}\overline{V}^{xy}-\overline{f_2}^{x}\overline{W}^{xz}+\frac{\overline{m}^{x}}{\overline{(\rho G^{\frac{1}{2}})}^{x}}\{\overline{V}^{xy}(U\partial_{2y}\overline{m}^{x}-\overline{V}^{xy}\partial_x m)-\frac{\overline{UW}^{xz}}{\alpha}\}+DIF.U, \tag{D2-1-19}$$

$$RV_{i,j,k+1/2}=\overline{f_1}^{y}\overline{W}^{yz}-\overline{f_3}^{y}\overline{U}^{xy}-\frac{\overline{m}^{y}}{\overline{(\rho G^{\frac{1}{2}})}^{y}}\{\overline{U}^{xy}(\overline{U}^{xy}\partial_y m-V\partial_{2x}\overline{m}^{y})+\frac{\overline{VW}^{yz}}{\alpha}\}+DIF.V, \tag{D2-1-20}$$

$$RW_{i,j,k+1/2}=f_2\overline{U}^{xz}-f_1\overline{V}^{yz}+\frac{m}{\overline{(\rho G^{\frac{1}{2}})}^{z}}\frac{(\overline{U}^{xy^2}+\overline{V}^{yz^2})}{\alpha}+DIF.W. \tag{D2-1-21}$$

and

$$u'=\frac{U}{\overline{(\rho G^{\frac{1}{2}})}^{x}},\ v'\frac{V}{\overline{(\rho G^{\frac{1}{2}})}^{y}},\ w'=\frac{W}{\overline{(\rho G^{\frac{1}{2}})}^{z}}. \tag{D2-1-22}$$

PFT in (C2-1-25) is computed directly from the virtual potential temperature.[1]

For (C2-1-26), the advection term of potential temperature is discretized as

$$ADV.\theta_{i,j,k}=\{m(\partial_x(U\overline{\theta}^{x})+\partial_y(V\overline{\theta}^{y}))+\partial_z(W^*\overline{\theta}^{z})\}\frac{m}{\rho G^{\frac{1}{2}}}\frac{\theta DIVT(U,V,W)}{\rho G^{\frac{1}{2}}}. \tag{D2-1-23}$$

At lateral boundaries, all the advection terms are computed by one-sided differences.

P.G. Advection terms for U, V and W are computed in sub.CADV4UV and sub.CADVC4W.

D-2-2 Higher order discretization for advection terms

For a function $f(x)$, $f(x\pm\Delta x)$ is expanded in the following Taylor series,

$$f(x\pm\Delta x)=f(x)\pm f'(x)\Delta x+f''(x)\frac{(\Delta x)^2}{2!}\pm f^{(3)}(x)\frac{(\Delta x)^3}{3!}+\cdots \tag{D2-2-2}$$

That is,

$$f(x+\Delta x)-f(x-\Delta x)=f'(x)2\Delta x+f^{(3)}(x)\frac{2(\Delta x)^3}{3!}+f^{(5)}(x)\frac{2(\Delta x)^5}{5!}+\cdots \tag{D2-2-3}$$

Rearranging (D2-2-3) gives

$$f'(x)_{\Delta x}=\frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}-f^{(3)}(x)\frac{(\Delta x)^2}{3!}-f^{(5)}(x)\frac{(\Delta x)^4}{5!}-\cdots \tag{D2-2-4}$$

The first term on the right-hand side is the second-order centered difference that contains errors of higher order

[1] In Ikawa and Saito's (1991) quasi-compressible model, *PFT* was defined by (C2-2-12) and discretized by

$$PFT=\frac{1}{g}\frac{\partial\overline{BUOY'}^{z}}{\partial t}.$$

In addition to the linearization, this expression contained errors due to double averaging of potential temperature because $BUOY'$ is defined at a half level.

than $(\Delta x)^2$. For the advection term, this error affects the phase speed of the quantity, which causes a dispersion.

Substituting $2\Delta x$ for Δx in (D2-2-4), we obtain

$$f'(x)_{2\Delta x}=\frac{f(x+2\Delta x)-f(x-2\Delta x)}{4\Delta x}-f^{(3)}(x)\frac{4(\Delta x)^2}{3!}-f^{(5)}(x)\frac{16(\Delta x)^4}{5!}-\cdots \qquad \text{(D2-2-5)}$$

Computing $f'(x)$ by $\{4\times(\text{D2-2-4})-(\text{D2-2-5})\}\div 3$ gives

$$f'(x)=\frac{3}{4}f'(x)_{\Delta x}-\frac{1}{3}f'(x)_{2\Delta x}$$

$$=\frac{-f(x+2\Delta x)+8f(x+\Delta x)-8f(x-\Delta x)+f(x-2\Delta x)}{12\Delta x}+f^{(5)}(x)\frac{4(\Delta x)^4}{5!}+\cdots \qquad \text{(D2-2-6)}$$

The first term on the right-hand side is the fourth-order centered difference derived by the four-point method.

Higher order advection terms of scalar variables are computed in advective form as

$$ADV.\theta_{i,j,k}=\{m(\overline{U}^x\partial_{x4}\theta+\overline{V}^z\partial_{y4}\theta)+\overline{W^*}^z\partial_{2z}\theta\}\frac{m}{\rho G^{\frac{1}{2}}}, \qquad \text{(D2-2-7)}$$

where, assuming uniform grid intervals, the fourth-order gradient is defined by

$$\partial_{x4}\theta=\frac{-\theta_{i+2,j,k}+8\theta_{i-1,j,k}+\theta_{i-2,j,k}}{12\Delta x}. \qquad \text{(D2-2-8)}$$

In the same manner, the third-order upstream gradient is defined by

$$\partial_{x3}\theta=\frac{2\theta_{i+1,j,k}+3\theta_{i,j,k}-6\theta_{i-1,j,k}+\theta_{i-2,j,k}}{6\Delta x} \quad for\overline{U}^x\geq 0,$$

$$\partial_{x3}=\frac{-\theta_{i+2,j,k}+6\theta_{i+1,j,k}-3\theta_{i,j,k}-2\theta_{i-1,j,k}}{6\Delta x} \quad for\overline{U}^x<0, \qquad \text{(D2-2-9)}$$

For the wind component, the fourth-order gradient (D2-2-8) is directly applied to (D2-1-16) - (D2-1-18), *e.g.,*

$$ADVU_{i+1/2,j,k}=\overline{m}^x\{\partial_{x4}(m\overline{\overline{U}^x u'}^x)+\partial_{y4}(m\overline{\overline{V}^x u'}^y)\}+\partial_z(m\overline{\overline{W^*}^x u'}^z)-\frac{u}{\overline{m}^x}PRC^x, \qquad \text{(D2-2-10)}$$

At the lateral boundaries, all the advection terms are computed by one-sided differences, and just inside the lateral boundaries, the advection terms are computed by the second order accuracy.

The fourth-order advection scheme described in this chapter was used for the GCSS CASE-1 model intercomparison, where a TOGA-COARE observed squall line was simulated (Redelsperger *et al.,* 1999). Currently, this option works for off-line ideal simulation but is instable for simulations in real situations. Further testing should be done to use a higher order scheme for real case simulation with nesting.

D-2-3 Modified centered difference scheme for advection

A new advection scheme has been developed to remove the numerical errors that the centered-difference advection scheme produces on the upstream side. These errors increase rapidly as the grid size decreases, which causes significant problems, especially in a high-resolution model with a horizontal grid size of less than 5 km. For positive prediction values, Smolarkiewicz (1983) and Hsu and Arakawa (1990) developed a more accurate advection scheme. Regardless of the sign of predicted values, the present scheme is designed so that values calculated by the centered-difference advection scheme lie between the maximum and minimum of those in the upstream neighboring grid boxes.

This new scheme is outlined for a two-dimensional case. The second-order centered-difference advection scheme for a predicted value F is given as

$$F_{i,j}^{m+1}=F_{i,j}^{m-1}-2\Delta t\left(\frac{U_{i+\frac{1}{2},j}^{m}F_{i+\frac{1}{2},j}^{m}-U_{i-\frac{1}{2},j}^{m}F_{i-\frac{1}{2},j}^{m}}{\Delta x}+\frac{V_{i,j+\frac{1}{2}}^{m}F_{i,j+\frac{1}{2}}^{m}-V_{i,j-\frac{1}{2}}^{m}F_{i,j-\frac{1}{2}}^{m}}{\Delta x}\right), \qquad \text{(D2-3-1)}$$

where the superscript m denotes the m-th time level, U and V are the momentum fluxes, Δx and Δy are the grid intervals in the x- and y-directions, and Δt is the time step. First, the rates of advection from the upstream side are calculated as

$$UP=Max(Min(-\frac{2\Delta tU_{i+\frac{1}{2},j}^{m}}{\Delta x},1.0),0.0),$$

$$UM=Max(Min(-\frac{2\Delta tU_{i-\frac{1}{2},j}^{m}}{\Delta x},1.0),0.0),$$

$$VP=Max(Min(-\frac{2\Delta tV_{i+\frac{1}{2},j}^{m}}{\Delta y},1.0),0.0),$$

$$VM=Max(Min(-\frac{2\Delta tV_{i-\frac{1}{2},j}^{m}}{\Delta y},1.0),0.0), \qquad \text{(D-2-3-2)}$$

and

$$V\max=Max(UP,UM,VP,VM).$$

Here, these rates are zero for the advection from the downstream side, and $V\max$ is the rate in the direction where the maximum advection is determined. The value in the neighboring grid box in this direction at the time level m-1 is denoted as $F0$. Next, the acceptable maximum Fx and the minimum Fn of the value considered by advection are determined as

$$Fx_{i,j}=Max(F_{i,j}^{m-1},\ F0+(_{i+1,j}^{m-1}-F0)\frac{UP}{V\max},F0+(F_{i,j+1}^{m-1}-F0)\frac{VP}{V\max},$$
$$F0+(_{i-1,j}^{m-1}-F0)\frac{UM}{V\max},F0+(F_{i,j-1}^{m-1}-F0)\frac{VM}{V\max},$$

and (D2-3-3)

$$Fn_{i,j}=Min(F_{i,j}^{m-1},\ F0+(_{i+1,j}^{m-1}-F0)\frac{UP}{V\max},F0+(F_{i,j+1}^{m-1}-F0)\frac{VP}{V\max},$$
$$F0+(_{i-1,j}^{m-1}-F0)\frac{UM}{V\max},F0+(F_{i,j-1}^{m-1}-F0)\frac{VM}{V\max},$$

When the value calculated by (D2-3-1) does not fall between Fx and Fn (*i.e.*, $F_{i,j}^{m+1}-Fx_{i,j}>0$ or $F_{i,j}^{m+1}-Fn_{i,j}<0$), it is replaced by either Fx or Fn. The difference produced by this replacement is distributed among the neighbor grid boxes as follows. The difference between the value calculated by (D2-3-1) and Fx (Fn) is denoted as

$$F1_{i,j}=-F_{i,j}^{m+1}+Fx_{i,j}(=F_{i,j}^{m+1}-Fn_{i,j}). \qquad \text{(D2-3-4)}$$

The total acceptable amount of the neighboring grid boxes FS is calculated as

$$FS_{i,j}=Max(F1_{i+1,j},0.0)+Max(F1_{i-1,j},0.0)+Max(F1_{i,j+1},0.0)+Max(F1_{i,j-1},0.0). \qquad \text{(D2-3-5)}$$

Therefore, the values of the neighboring grid boxes in which the value calculated by (D2-3-1) is between Fx and Fn are adjusted as

$$R=\frac{Max(FS_{i,j}+F1_{i,j},\,0.0)}{F1_{i,j}},$$

$$F1^{*}_{i\pm1,j}=R\times F1_{i\pm1,j},\quad for \quad F1_{i\pm1,j}>0,$$

and (D2-3-6)

$$F1^{*}_{i,j\pm1}=R\times F1_{i,j\pm1},\quad for \quad F1_{i,j\pm1}>0,$$

where $F1^{*}_{i,j,}$ is the adjusted value of $F1_{i,j}$. By substituting $F1^{*}_{i,j}$ into the left-hand side of (D2-3-4), we can calculate the adjusted value of $F^{m+1}{}_{i,j}$. For $FS_{i,j}+F1_{i,j}<0$, the total amount of predicted values cannot be exactly preserved, but the validation test of this scheme indicates that this error is very small (see Fig. 11.12 in Saito and Kato, 1999).

D-3. Pressure equation solver

The basic concept of the pressure equation solver on a variable grid was reviewed in B-6 of Ikawa and Saito (1991), but the expression was simplified assuming $G^{1/2}=1$. In this technical report, we describe the details of the pressure solver of HI-VI again following the programming code including the map factor.

D-3-1 Unified expression of the pressure equation

As shown in chapter C-3, the pressure equations take the following form for the E-HI-VI scheme

$$\alpha_{HI}\left(\frac{\partial^2\Delta^2 P}{\partial x^2}+\frac{\partial^2\Delta^2 P}{\partial y^2}\right)+\frac{\partial^2\Delta^2 P}{\partial z^2}+\frac{\partial}{\partial z}(h_{HI}\Delta^2 P)+e_{HI}\Delta^2 P=FP.HI, \tag{D3-1-1}$$

where

$$\alpha_{HI}=\frac{\overline{\overline{m^2}}}{\overline{\overline{\frac{m}{G^{\frac{1}{2}}}}}\,\overline{\overline{\frac{1}{mG^{\frac{1}{2}}}}}}, \tag{D3-1-2}$$

$$h_{HI}=\frac{1}{\overline{\overline{\frac{1}{mG^{\frac{1}{2}}}}}}\,\overline{\overline{\frac{g}{mC^2_m}}}, \tag{D3-1-3}$$

$$e_{HI}=-\frac{1}{\overline{\overline{\frac{m}{G^{\frac{1}{2}}}}}\,\overline{\overline{\frac{1}{mG^{\frac{1}{2}}}}}}\,\frac{1}{\overline{\overline{C^2_m}}(1+\alpha)^2(\Delta t)^2}, \tag{D3-1-4}$$

$$FP.HI=\frac{1}{\overline{\overline{\frac{m}{G^{\frac{1}{2}}}}}\,\overline{\overline{\frac{1}{mG^{\frac{1}{2}}}}}}\left\{\frac{ADVP'}{(1+\alpha)\Delta t}DIVS(ADVU',ADVV',ADVW')\right\}, \tag{D3-1-5}$$

For the AE scheme,

$$\alpha_{AE}\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right)+\frac{\partial^2 P}{\partial z^2}+\frac{\partial}{\partial z}(h_{AE}P)=FP.AE.INV+FP.AE.VAR, \tag{D3-1-6}$$

where

$$\alpha_{AE}=(\overline{\overline{G^{\frac{1}{2}}}})^2, \tag{D3-1-7}$$

$$h_{AEI}=\frac{g}{\overline{\overline{C^2_s}}}\overline{\overline{G^{\frac{1}{2}}}}\approx\frac{\overline{\overline{gG^{\frac{1}{2}}}}}{C^2_s}, \tag{D3-1-8}$$

$$FP.AE.INV = -\overline{\overline{(G^{\frac{1}{2}})}}^2 DIVT(ADVU-RU, ADVV-RV, ADVW-RW-BUOY')$$

$$+\frac{\overline{\overline{(G^{\frac{1}{2}})}}^2}{2\Delta t} DIVT(U^{\tau-1}, V^{\tau-1}, W^{\tau-1}), \qquad \text{(D3-1-9)}$$

$$FP.AE.VAR = -\overline{\overline{(G^{\frac{1}{2}})}}^2 DIVR\left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \overline{\overline{\frac{1}{G^{\frac{1}{2}}}}}\frac{\partial P}{\partial z^*} + \frac{P}{C_s^2}g\right)$$

$$-\overline{\overline{(G^{\frac{1}{2}})}}^2 DIVT\left\{\frac{\partial G^{\frac{1}{2}}G^{13}P}{G^{\frac{1}{2}}\partial z^*}, \frac{\partial G^{\frac{1}{2}}G^{23}P}{G^{\frac{1}{2}}\partial z^*}, \left(\frac{1}{G^{\frac{1}{2}}}\frac{\partial}{\partial z^*} + \frac{g}{C_s^2} - \overline{\overline{\frac{1}{G^{\frac{1}{2}}}}}\frac{\partial}{\partial z^*} - \frac{g}{C_s^2}\right)P\right\}. \qquad \text{(D3-1-10)}$$

For the E-HE-VI scheme, the pressure equations are given by (C3-2-11) to (C3-2-13).

Here, (D3-1-1) is the three-dimensional Helmholtz equation for $\Delta^2 P$, (D3-1-6) is the three-dimensional Poisson equation for P, and (C3-2-11) is the one-dimensional Helmholtz equation for P^{β}. Following Ikawa and Saito (1991), the 3-D elliptic equations can be reduced to a one-dimensional elliptic equation by the Dimension Reduction Method, which is described in D-3-3.

D-3-2 Finite discretization form for the pressure equation

In the finite discretization of second-order accuracy, the centered differences may be written as

$$\frac{\partial}{\partial x}(\phi)_j = (\phi_x)_j = \frac{1}{\Delta x_j}\{(\phi)_{j+\frac{1}{2}} - (\phi)_{j-\frac{1}{2}}\}, \qquad \text{(D3-2-1)}$$

$$\frac{\partial^2}{\partial x^2}(\phi)_j = \frac{1}{\Delta x_j}\{(\phi_x)_{j+\frac{1}{2}} - (\phi_x)_{j-\frac{1}{2}}\}$$

$$= \frac{1}{\Delta x_j}\left\{\frac{(\phi)_{j+1} - (\phi)_j}{\Delta x_{j+\frac{1}{2}}} - \frac{(\phi)_j - (\phi)_{j-1}}{\Delta x_{j-\frac{1}{2}}}\right\}$$

$$= \frac{(\phi)_{j+1}}{\Delta x_j \Delta x_{j+\frac{1}{2}}} - \frac{(\phi)_j}{\Delta x_j}\left(\frac{1}{\Delta x_{j+\frac{1}{2}}} + \frac{1}{\Delta x_{j-\frac{1}{2}}}\right) + \frac{(\phi)_{j-1}}{\Delta x_j \Delta x_{j-\frac{1}{2}}}. \qquad \text{(D3-2-2)}$$

Thus, (D3-1-1) and (D3-1-6) may be discretized for a grid point (i, j, k) as

$$a\left\{\frac{P_{i,j,k}}{\Delta x_i \Delta x_{i+\frac{1}{2}}} - \frac{P_{i,j,k}}{\Delta x_j}\left(\frac{1}{\Delta x_{i+\frac{1}{2}}} + \frac{1}{\Delta x_{i-\frac{1}{2}}}\right) + \frac{P_{i-1,j,k}}{\Delta x_i \Delta x_{i-\frac{1}{2}}}\right.$$

$$+\frac{P_{i,j+1,k}}{\Delta y_j \Delta y_{j+\frac{1}{2}}} - \frac{P_{i,j,k}}{\Delta y_j}\left(\frac{1}{\Delta y_{j+\frac{1}{2}}} + \frac{1}{\Delta y_{j-\frac{1}{2}}}\right) + \frac{P_{i,j-1,k}}{\Delta y_j \Delta y_{j-\frac{1}{2}}}$$

$$+\frac{1}{\Delta z_k}\left\{\frac{1}{\Delta z_{i+1/2}}(P_{i,j,k+1} - P_{i,j,k}) - \frac{1}{\Delta z_{k-1/2}}(P_{i,j,k-1})\right\}$$

$$+\frac{1}{\Delta z_k}\left(h_{i,j,k+\frac{1}{2}}\frac{P_{i,j,k+1}P_{i,j,k}}{2} - h_{i,j,k-\frac{1}{2}}\frac{P_{i,j,k}+P_{i,j,k-1}}{2}\right)$$

$$+eP_{i,j,k} = F_{i,j,k}. \qquad \text{(D3-2-3)}$$

Here, h is given at a half level. In the above expression, $\Delta^2 P$ in (D3-1-1) is replaced with P and subscripts are omitted for a unified description.

At the boundaries, we assume the following Neumann-type boundary conditions.

$$\frac{\partial P}{\partial x} = B_x, \qquad \text{(D3-2-4)}$$

$$\frac{\partial P}{\partial y}=B_y, \tag{D3-2-5}$$

$$\frac{\partial P}{\phi z}+hP=B_z. \tag{D3-2-6}$$

The detailed formulation of B is discussed in F-1. In the finite discretization form, the above equations are written as

$$\frac{1}{\Delta x_{1+\frac{1}{2}}}(P_{2,j,k}-P_{1,j,k})=B_{1,j,k},$$

$$\frac{1}{\Delta x_{nx-\frac{1}{2}}}(P_{nx,j,k}-P_{nx-1,j,k})=B_{nx,j,k}, \tag{D3-2-7}$$

$$\frac{1}{\Delta y_{1+\frac{1}{2}}}(P_{i,2,k}-P_{i,1,k})=B_{i,1,k},$$

$$\frac{1}{\Delta y_{nx+\frac{1}{2}}}(P_{i,ny,k}-P_{i,ny-1,k})=B_{i,ny,k}, \tag{D3-2-8}$$

$$\frac{1}{\Delta z_{1+1/2}}(P_{i,j,2}-P_{i,j,1})+\frac{h_{i,j,1+\frac{1}{2}}}{2}(P_{i,j,2}+P_{i,j,1})=B_{i,j,1},$$

$$\frac{1}{\Delta z_{nz-1/2}}(P_{i,j,nz}-P_{i,j,nz-1})+\frac{h_{i,j,nz-\frac{1}{2}}}{2}(P_{i,j,nz}+P_{i,j,nz-1})=B_{i,j,nz}. \tag{D3-2-9}$$

Here, (nx, ny, nz) is the model dimension, and the boundary values $B_{1,j,k}$, $B_{nx,j,k}$, $B_{i,1,k}$, $B_{i,ny,k}$, $B_{i,j,1}$, $B_{i,j,nz}$ are given at the locations $x=1+1/2$, $nx-1/2$, $y=1+1/2$, $ny-1/2$, $z=1+1/2$, $nz-1/2$, respectively.

Incorporating the boundary condition (D3-2-7), the finite discretization in the x-direction of the first term on the left-hand side of (D3-2-3) is expressed as

$$\alpha Y_A^{-1}A\Pi_{,j,k}=\Phi_{j,k}, \tag{D3-2-10}$$

for i running from 2 to nx-1. Here,

$$Y_A^{-1}=\begin{pmatrix} \frac{1}{\Delta x_2} & 0 & & \\ 0 & \frac{1}{\Delta x_3} & 0 & \\ \cdots & \cdots & & \cdots \\ 0 & \frac{1}{\Delta x_{nx-2}} & & 0 \\ & 0 & & \frac{1}{\Delta x_{nx-1}} \end{pmatrix}, \tag{D3-2-11}$$

$$A=\begin{pmatrix} -\frac{1}{\Delta x_{2+\frac{1}{2}}} & \frac{1}{\Delta x_{2+\frac{1}{2}}} & & & \\ \frac{1}{\Delta x_{3-\frac{1}{2}}} & -(\frac{1}{\Delta x_{3+\frac{1}{2}}}+\frac{1}{\Delta x_{3-\frac{1}{2}}}) & \frac{1}{\Delta x_{3+\frac{1}{2}}} & & \\ & \cdots & \cdots & \cdots & \\ & & \frac{1}{\Delta x_{nx-2+\frac{1}{2}}} & -(\frac{1}{\Delta x_{nx-2+\frac{1}{2}}}+\frac{1}{\Delta x_{nx-2-\frac{1}{2}}}) & \frac{1}{\Delta x_{nx-2+\frac{1}{2}}} \\ & & & \frac{1}{\Delta x_{nx-1-\frac{1}{2}}} & -\frac{1}{\Delta x_{nx-1-\frac{1}{2}}} \end{pmatrix}, \tag{D3-2-12}$$

$$\Pi_{,j,k}=\begin{pmatrix} P_{2,j,k} \\ P_{3,j,k} \\ \cdots \\ P_{nx-2,j,k} \\ P_{nx-1,j,k} \end{pmatrix}, \tag{D3-2-13}$$

$$\Phi_{j,k}=\begin{pmatrix} F_{2,j,k} \\ F_{3,j,k} \\ \cdots \\ F_{nz-2,j,k} \\ F_{nz-1,j,k} \end{pmatrix}, \tag{D3-2-14}$$

and

$$\begin{aligned} F'_{2,j,k} &= F_{2,j,k}+aB_{1,j,k}/\Delta x_2, \\ F'_{nx-1,j,k} &= F_{nx-1,j,k}-aB_{nx,j,k}/\Delta x_{nx-1}. \end{aligned} \tag{D3-2-15}$$

For a uniform grid, the product of (D3-2-11) and (D3-2-12) is simply rewritten as

$$Y_A^{-1}A=A_u=\left(\frac{1}{\Delta x}\right)^2\begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \cdots & \cdots & \cdots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix}, \tag{D3-2-16}$$

If the boundary condition is cyclic, the matrix has the following form

$$A_c=\left(\frac{1}{\Delta x}\right)^2\begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \cdots & \cdots & \cdots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix}, \tag{D3-2-17}$$

with $B_{1,j,k}$ and $B_{nx,j,k}$ vanishing in (D3-2-15).

Under Dirichlet-type boundary conditions (not implemented in the model yet), (D3-2-10) becomes

$$a\left(\frac{1}{\Delta x}\right)^2\begin{pmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \cdots & \cdots & \cdots & \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{pmatrix}\begin{pmatrix} P_{2,j,k} \\ P_{3,j,k} \\ \cdots \\ P_{nx-2,j,k} \\ P_{nx-1,j,k} \end{pmatrix}=\begin{pmatrix} F_{2,j,k}-a\dfrac{P_{1,j,k}}{(\Delta x)^2} \\ F_{3,j,k} \\ \cdots \\ F_{nx-2,j,k} \\ F_{nx-1,j,k}-a\dfrac{P_{nx,j,k}}{(\Delta x)^2} \end{pmatrix} \tag{D3-2-18}$$

Hereafter, we use $njx=nx-2$, $njy=ny-2$ for expression. Since (D3-2-3) is separable in the x- and y-directions, applying the same manner in the y-direction yields

$$a\begin{pmatrix} A_u & & & & \\ & A_u & & & \\ & & \cdots & & \\ & & & A_u & \\ & & & & A_u \end{pmatrix}\begin{pmatrix} \Pi_{,2,k} \\ \Pi_{,3,k} \\ \cdots \\ \Pi_{,ny-2,k} \\ \Pi_{,ny-1,k} \end{pmatrix}+a\left(\frac{1}{\Delta y}\right)^2\begin{pmatrix} -I_{njx} & I_{njx} & & & \\ I_{njx} & -2I_{njx} & I_{njx} & & \\ & \cdots & \cdots & \cdots & \\ & & I_{njx} & -2I_{njx} & I_{njx} \\ & & & I_{njx} & -I_{njx} \end{pmatrix}\begin{pmatrix} \Pi_{,2,k} \\ \Pi_{,3,k} \\ \cdots \\ \Pi_{,ny-2,k} \\ \Pi_{,ny-1,k} \end{pmatrix}=\begin{pmatrix} \Phi_{2,k}' \\ \Phi_{3,k} \\ \cdots \\ \Phi_{ny-2,k} \\ \Phi_{ny-1,k}' \end{pmatrix} \tag{D3-2-19}$$

for each level discretization. Here, I_{njx} is a unit matrix whose dimensions are $(njx,\ njx)$. Introducing a tensor product operator for $(m,\ m)$ matrix M and $(n,\ n)$ matrix N

$$M \otimes N \equiv \begin{pmatrix} m_{11}N & m_{12}N & \cdots & m_{1m}N \\ m_{21}N & m_{22}N & & m_{2m}N \\ & & & \\ m_{m1}N & & & m_{mm}N \end{pmatrix} \tag{D3-2-20}$$

(D3-2-19) can be written in the following form

$$a(I_{njy} \otimes Y_A^{-1}A + Y_B^{-1}B \otimes I_{njx})\Pi_{,,k} = \Phi_{,,k}. \tag{D3-2-21}$$

Here, I_{njy} is a unit matrix of (njy, njy), B is a matrix of (njy, njy) that is similar to A, and $\Pi_{,,k}$ and $\Phi_{,,k}$ are $(njx * njy, 1)$ matrices consisting of $P_{,j,k}$ and $F_{,j,k}$ (incorporating the boundary conditions).

Finally, the finite discretization (D3-2-3) is written in the following form

$$a(I_{njy} \otimes Y_A^{-1}A + Y_B^{-1}B \otimes I_{njx})\Pi_{,,k} + r_k\Pi_{,,k-1} + (s_k + e_k)\Pi_{,,k} + t_k\Pi_{,,k+1} = \Phi_{,,k}, \tag{D3-2-22}$$

for $2 \leqq k \leqq nz-1$. Here,

$$r_k = \frac{1}{\Delta z_k}\left(\frac{1}{\Delta z_{k-\frac{1}{2}}} - \frac{h_{k-\frac{1}{2}}}{2}\right),$$

$$s_k = \frac{1}{\Delta z_k}\left(\frac{1}{\Delta z_{k+\frac{1}{2}}} - \frac{1}{\Delta z_{k-\frac{1}{2}}} + \frac{h_{k+\frac{1}{2}} - h_{k-\frac{1}{2}}}{2}\right),$$

$$t_k = \frac{1}{\Delta z_k}\left(\frac{1}{\Delta z_{k+\frac{1}{2}}} + \frac{h_{k+\frac{1}{2}}}{2}\right). \tag{D3-2-23}$$

The upper and lower boundary conditions (D3-2-9) are rewritten as

$$\left(-\frac{1}{\Delta z_{1+\frac{1}{2}}} + \frac{h_{1+\frac{1}{2}}}{2}\right)\Pi_{,,1} + \left(\frac{1}{\Delta z_{1+\frac{1}{2}}} + \frac{h_{1+\frac{1}{2}}}{2}\right)\Pi_{,,2} = \Phi_{,,1},$$

$$\left(-\frac{1}{\Delta z_{nz-\frac{1}{2}}} + \frac{h_{nz-\frac{1}{2}}}{2}\right)\Pi_{,,nz-1} + \left(\frac{1}{\Delta z_{nz-\frac{1}{2}}} + \frac{h_{nz-\frac{1}{2}}}{2}\right)\Pi_{,,nz} = \Phi_{,,nz}. \tag{D3-2-24}$$

Here, $\Phi_{,,1}$ and $\Phi_{,,nz}$ are $(njx * njy, 1)$ matrices consisting of $B_{i,j,1}$ and $B_{i,j,nz}$. These equations are rewritten as

$$t_1\Pi_{,,2} + s_1\Pi_{,,1},$$
$$r_{nz}\Pi_{,,nz-1} + s_{nz}\Pi_{,,nz} = \Phi_{,,nz}. \tag{D3-2-25}$$

D-3-3 Dimension Reduction Method

The dimension reduction method described in Ikawa and Saito (1991) is used to solve (D3-2-22). This method projects the pressure equation onto the horizontal eigen space, and leaves a one-dimensional vertical equation. Here, we focus primarily on a uniform grid (D3-2-16) since the usual computation is done with a uniform grid. A detailed description of the dimension reduction method for a variable grid is presented in Ikawa and Saito (1991).

For a uniform grid, the eigen value matrix and eigen function for A_u (D3-2-16) are given as

$$\Lambda_a = \frac{1}{(\Delta x)^2}\begin{pmatrix} 0 & 0 & & & \\ 0 & \cdots & 0 & & \\ & & -4\sin^2\frac{k-1}{2njx}\pi & & \\ & & 0 & \cdots & 0 \\ 0 & & & 0 & -4\sin^2\frac{njx-1}{2njx}\pi \end{pmatrix}, \tag{D3-3-1}$$

and

$$P=\frac{\sqrt{2}}{\sqrt{njx}}\begin{pmatrix} \frac{1}{\sqrt{2}} & \cdots & \cos\frac{(k-1)(1-1/2)}{njx}\pi & \cdots & \cos\frac{(njx-1)(1-1/2)}{njx}\pi \\ \cdots & & \cdots & & \\ \frac{1}{\sqrt{2}} & \cdots & \cos\frac{(k-1)(m-1/2)}{njx}\pi & \cdots & \cos\frac{(njx-1)(m-1/2)}{njx}\pi \\ \cdots & & \cdots & & \\ \frac{1}{\sqrt{2}} & \cdots & \cos\frac{(k-1)(njx-1/2)}{njx}\pi & \cdots & \cos\frac{(njx-1)(njx-1/2)}{njx}\pi \end{pmatrix}. \quad \text{(D3-3-2)}$$

Under the cyclic boundary condition, as described in Ikawa (1981), they are

$$\Lambda_a=\frac{1}{(\Delta x)^2}\begin{pmatrix} 0 & & & & & & \\ & \cdots & & & & & \\ & & 4\sin^2\frac{k-1}{njx}\pi & & & & \\ & & & -4\sin^2\frac{\pi}{2} & & & \\ & & & & -4\sin^2\frac{\pi}{2} & & \\ & & & & & \cdots & \\ & & & & & -\sin^2\frac{2(k-njx/2-1)}{njx}\pi & \\ & & & & & & 4\sin^2\frac{njx-1}{2njx}\pi \end{pmatrix}, \quad \text{(D3-3-3)}$$

$$P=\frac{\sqrt{2}}{\sqrt{njx}}\begin{pmatrix} \frac{1}{\sqrt{2}} & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2}} & \cos\frac{2\pi}{njx} & \cdots & \cos\frac{2\pi}{njx}\frac{njx}{2} & \sin\frac{2\pi}{njx} & \cdots & \sin\frac{2\pi}{njx}(\frac{njx}{2}-1) \\ \cdots & \cdots & & \cdots & \cdots & & \cdots \\ \frac{1}{\sqrt{2}} & \cos\frac{2\pi}{njx}m & \cdots & \cos\frac{2\pi}{njx}\frac{njx}{2}m & \sin\frac{2\pi}{njx}m & & \sin\frac{2\pi}{njx}(\frac{njx}{2}-1)m \\ \cdots & \cdots & & \cdots & \cdots & & \\ \frac{1}{\sqrt{2}} & \cos\frac{2\pi}{njx}(njx-1) & \cdots & \cos\frac{2\pi}{njx}\frac{njx}{2}(njx-1) & \sin\frac{2\pi}{njx}(njx-1) & & \sin\frac{2\pi}{njx}(\frac{njx}{2}-1)(njx-1) \end{pmatrix}, \quad \text{(D3-3-4)}$$

respectively. For the Diriclet boundary condition, see Ogura (1969).

Using the following property of a tensor product

$$(U_1\otimes U_2)(U_3\otimes U_4)=(U_1U_3)\otimes(U_2U_4). \quad \text{(D3-3-5)}$$

and operating $Q^{-1}\otimes P^{-1}$, the coefficients of $\Pi_{,,k}$ in (D3-2-22) become

$$\begin{aligned} &Q^{-1}\otimes P^{-1}(I_{njy}\otimes A_u+B_u\otimes I_{njx})(Q\otimes P)Q^{-1}\otimes P^{-1} \\ &=(Q^{-1}I_{njy}Q\otimes P^{-1}A_uP+Q^{-1}B_uQ\otimes P^{-1}I_{njx}P)Q^{-1}\otimes P^{-1} \\ &=(I_{njy}\otimes\Lambda_a+\Lambda_b\otimes I_{njz})Q^{-1}\otimes P^{-1}, \end{aligned} \quad \text{(D3-3-6)}$$

where Q and Λ_b are the eigen function and eigen value matrix of B_u.

Thus, (D3-2-22) becomes

$$\begin{aligned} &a(I_{njy}\otimes\Lambda_u+\Lambda_b\otimes I_{njx})Q^{-1}\otimes P^{-1}\Pi_{,,k} \\ &+r_kQ^{-1}\otimes P^{-1}\Pi_{,,k-1}+(s_k+e_k)Q^{-1}\otimes P^{-1}\Pi_{,,k} \\ &+t_kQ^{-1}\otimes P^{-1}\Pi_{,,k+1}=Q^{-1}\otimes P^{-1}\Phi_{,,k}. \end{aligned} \quad \text{(D3-3-7)}$$

If we define

$$\hat{\Pi}_{,,k} \equiv Q^{-1} \otimes P^{-1} \Pi_{,,k},$$
$$\hat{\Phi}_{,,k} \equiv Q^{-1} \otimes P^{-1} \Phi_{,,k}, \qquad \text{(D3-3-8)}$$

(D3-3-7) becomes

$$r_k \hat{\Pi}_{,,k-1} + \{a(I_{njy} \otimes \Lambda_a + \Lambda_b \otimes I_{njx}) + s_k + e_k\} \hat{\Pi}_{,,k} + t_k \hat{\Pi}_{,,k+1} = \Phi_{,,k}. \qquad \text{(D3-3-9)}$$

That is, for $\hat{\Phi}_{i,j,k}$

$$\begin{pmatrix} s_1 & t_1 & & & & \\ r_2 & s_2+e_2+a(\lambda a_i+\lambda b_j) & t_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & r_k & s_k+e_k+a(\lambda a_i+\lambda b_j) & t_k & \\ & & & \cdots & \cdots & \\ & & & & r_{nz} & s_{nz} \end{pmatrix} \begin{pmatrix} \hat{\Pi}_{i,j,1} \\ \hat{\Pi}_{i,j,2} \\ \cdots \\ \hat{\Pi}_{i,j,k} \\ \cdots \\ \hat{\Pi}_{i,j,nz} \end{pmatrix} = \begin{pmatrix} \hat{\Phi}_{i,j,1} \\ \hat{\Phi}_{i,j,2} \\ \cdots \\ \hat{\Phi}_{i,j,k} \\ \cdots \\ \hat{\Phi}_{i,j,nz} \end{pmatrix}. \qquad \text{(D3-3-10)}$$

where λa_i and λb_j are i-th and j-th components of Λ_a and Λ_b. This is a one-dimensional Helmholtz equation, similar to (C3-2-11). When $\hat{\Pi}_{,,k}$ is given, $\Pi_{,,k}$ can be obtained by inverse transform

$$\Pi_{,,k} = Q \otimes P \hat{\Pi}_{,,k}. \qquad \text{(D3-3-11)}$$

D-3-4 Pressure equation solver with the Gaussian elimination

(D3-3-10) can be solved by the Gaussian elimination if the (nz, nz) coefficients matrix is not singular. For simplicity, we omit suffixes i,j from the expression, and rewriting $s_k + e_k + a(\lambda a_i + \lambda b_j)$ as s_k for $2 \leqq k \leqq nz-1$ and express the coefficients of the tri-diagonal matrix as

$$\begin{pmatrix} s_1 & t_1 & & & & \\ r_g & s_2 & t_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & r_k & s_k & t_k & \\ & & & \cdots & \cdots & \\ & & & & r_{nz} & s_{nz} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \\ p_k \\ \\ p_{nz} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \\ f_k \\ \\ f_{nz} \end{pmatrix}. \qquad \text{(D3-4-1)}$$

Multiplying the first line of (D3-4-1) by $a_1 = 1/s_1$, and setting $b_1 = -t_1 a_1$, $g_1 = f_1 a_1$ yields

$$\begin{pmatrix} 1 & -b_1 & & & & \\ r_2 & s_2 & t_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & r_k & s_k & t_k & \\ & & & \cdots & \cdots & \\ & & & & r_{nz} & s_{nz} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \\ p_k \\ \\ p_{nz} \end{pmatrix} = \begin{pmatrix} g_1 \\ f_2 \\ \\ f_k \\ \\ f_{nz} \end{pmatrix}. \qquad \text{(D3-4-2)}$$

Subtracting $r_2 \times$(first line) from the second line and multiplying by $a_2 = 1/(s_2 + r_2 b_1)$ gives

$$\begin{pmatrix} 1 & -b_1 & & & & \\ 0 & 1 & -b_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & r_k & s_k & t_k & \\ & & & \cdots & \cdots & \\ & & & & r_{nz} & s_{nz} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \\ p_k \\ \\ p_{nz} \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \\ f_k \\ \\ f_{nz} \end{pmatrix}, \qquad \text{(D3-4-3)}$$

where $b_2 = -t_2 a_2$, $g_2 = (f_2 - r_2 g_1) a_2$. In the same manner, by setting $a_k = 1/(s_k + r_k b_{k-1})$, $b_k = -t_k a_k$, $g_k = (f_k - r_k g_{k-1}) a_k$, the above formula becomes

$$\begin{pmatrix} 1 & -b_1 & & & & \\ & \cdots & \cdots & & & \\ & & 1 & -b_k & & \\ & & & \cdots & \cdots & \\ & & & & 1 & -b_{nz-1} \\ & & & & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ \\ p_k \\ \\ p_{nz-1} \\ p_{nz} \end{pmatrix} = \begin{pmatrix} g_1 \\ \\ g_k \\ \\ g_{nz-1} \\ g_{nz} \end{pmatrix}. \quad \text{(D3-4-4)}$$

The solution is given by $p_{nz}=g_{nz}$, ..., $p_k=g_k+b_kp_{k+1}$, ... $p_1=g_1+b_1p_2$. The coefficients a_k and b_k can be prepared in advance since they are independent of p_k or f_k.

As seen in (D3-3-1), $\lambda a_1=\lambda b_1=0$ for $(i, j)=(1, 1)$, thus, for an anelastic model ($e_k=0$), from (D3-2-23) to (D3-2-25), the following relation is satisfied:

$$(1 \quad \Delta z_2 \quad \cdots \quad \Delta z_k \quad \cdots \quad 1) \begin{pmatrix} s_1 & t_1 & & & & \\ r_2 & s_2 & t_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & r_k & s_k & t_k & \\ & & & \cdots & \cdots & \\ & & & & r_{nz} & s_{nz} \end{pmatrix} = 0, \quad \text{(D3-4-5)}$$

which means the (nz, nz) coefficient matrix becomes singular. As mentioned in Ikawa and Saito (1991), this corresponds to the non-uniqueness of the horizontal averaged pressure field in the anelastic model. To avoid this situation, the constraint

$$\hat{\Pi}_{1,1,nz-1}+\hat{\Pi}_{1,1,nz}=0, \quad \text{(D3-4-6)}$$

is imposed. In our case, we set $r_{nz}=s_{nz}=1$ for $(i, j)=(1, 1)$, which means that the horizontal averaged pressure in the anelastic model takes the same value of the reference atmosphere at the upper boundary.

P.G. The boundary conditions for pressure equations are set in sub.CFPBDV, and the forcing terms are set in sub.CPFORI and sub.CPFRV1. Eigen vector functions are set in sub.INIVG1 for a variable grid and sub.INIVG2 for a uniform grid. The Helmholtz equation is solved in sub.VHELMH where the Gaussian elimination is done in TRIDGH.