C. Model equations

C-1. Governing equations

C-1-1 Basic equations in Cartesian coordinates

For dry air, the atmospheric state is described by six fundamental variables; pressure p, density ρ, temperature T, and three wind components u, v and w. These six variables are governed by the following six equations.

a) Momentum equations

$$\frac{du}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial x}=dif.u, \qquad \text{(C1-1-1)}$$

$$\frac{dv}{dt}+\frac{1}{\rho}\frac{\rho p}{\partial y}=dif.v, \qquad \text{(C1-1-2)}$$

$$\frac{dw}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial z}+g=dif.w. \qquad \text{(C1-1-3)}$$

b) Continuity equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}+\frac{\partial\rho w}{\partial z}=0. \qquad \text{(C1-1-4)}$$

c) Thermodynamic equation

$$\frac{d\theta}{dt}=\frac{Q}{C_p\pi}+dif.\theta. \qquad \text{(C1-1-5)}$$

d) State equation

$$p=\rho RT. \qquad \text{(C1-1-6)}$$

Here, $dif.\phi$ stands for the diffusion term for field variable ϕ. Q is the diabatic heating rate, C_p the specific heat of dry air at constant pressure, and R the gas constant for dry air.

θ is the potential temperature, and π the Exner function defined as

$$\pi\equiv\left(\frac{p}{p_0}\right)^{R/C_p}, \qquad \text{(C1-1-7)}$$

$$\theta\equiv\frac{T}{\pi}. \qquad \text{(C1-1-8)}$$

For moist air, taking account of the partial pressure of water vapor, the state equation is replaced by

$$p=\rho_a RT_v, \qquad \text{(C1-1-9)}$$

where ρ_a is the density of air, which is the sum of the density of dry air and water vapor, and T_v is the virtual temperature defined as

$$T_v\equiv(1+0.61q_v)\,T. \qquad \text{(C1-1-10)}$$

Here, q_v is the mixing ratio of water vapor. Virtual potential temperature is defined in the same manner as in (C1-1-8) by

$$\theta_v\equiv\frac{T_v}{\pi}=\frac{(1+0.61q_v)\,T}{\pi}=(1+0.61q_v)\,\theta. \qquad \text{(C1-1-11)}$$

Using (C1-1-7) and (C1-1-8), the state equation (C1-1-9) is rewritten in the following formula.

$$\rho_a=\frac{p_0}{R\theta_v}\left(\frac{p}{p_0}\right)^{C_v/C_p}. \qquad \text{(C1-1-12)}$$

C-1-2 Mass-virtual potential temperature

a) Definition of the density and the state equation

When water substances exist, we define the density as the sum of the masses of moist air and the water substances per unit volume as

$$\rho \equiv \rho_d+\rho_v+\rho_c+\rho_r+\rho_i+\rho_s+\rho_g$$
$$=\rho_a+\rho_c+\rho_r+\rho_i+\rho_s+\rho_g, \qquad \text{(C1-2-1)}$$

where subscripts c, r, i, s, g stand for the cloud water, rain, cloud ice, snow, and graupel. ρ_d is the density of dry air and ρ_v, that of water vapor. In terms of the mixing ratio q, we can express the above formula as

$$\rho=\rho_d(1+q_v+q_c+q_r+q_i+q_s+q_g)$$
$$\cong\rho_a(1+q_c+q_r+q_i+q_s+q_g). \qquad \text{(C1-2-2)}$$

In the second expression, q is not the *mixing ratio* but the *specific value,* which is defined by the ratio of the mass of water substance to moist air. In this technical report, we neglect the difference between the two technical terms, following a custom in mesoscale numerical modeling (this is a matter of terminology rather than approximation).

As q is sufficiently small compared with unity (on the order of 10^{-3}), we can approximate (C1-2-2) with sufficient accuracy as

$$\rho_a=\rho(1+q_c+q_r+q_i+q_s+q_g)^{-1}$$
$$\cong\rho(1-q_c-q_r-q_i-q_s-q_g). \qquad \text{(C1-2-3)}$$

Considering that the state equation is an equation for substances in gas phase, we obtain the following equation by substituting (C1-2-3) into the left-hand side of (C1-1-12),

$$\rho=\frac{p_0}{R\theta_v}\left(\frac{p}{p_0}\right)^{C_v/C_p}(1-q_c-q_r-q_i-q_s-q_g)^{-1}$$
$$=\frac{p_0}{R\theta_m}\left(\frac{p}{p_0}\right)^{C_v/C_p}. \qquad \text{(C1-2-4)}$$

Here, θ_m is the *mass-virtual potential temperature* defined by

$$\theta_m\equiv\theta_v(1-q_c-q_r-q_i-q_s-q_g)$$
$$=\theta(1+0.61q_v)(1-q_c-q_r-q_i-q_s-q_g). \qquad \text{(C1-2-5)}$$

This quantity was introduced in Ikawa and Saito (1991) to expand the fundamental equations, but was not used in the actual programming. In this model, we use (C1-2-4) and compute buoyancy directly and exactly by the perturbation of density.

b) Continuity equation

When we define the density by (C1-2-4), we must consider the fall-out of water substances. We regard the rain, snow and graupel as the precipitable water substances and neglect the fall-out of the cloud water and cloud ice. Using mass-weighted bulk terminal velocity V, the time tendency of the density can be expressed as

$$\frac{\partial\rho}{\partial t}=-\left(\frac{\partial\rho u}{\partial x}+\frac{\partial\rho u}{\partial y}-\frac{\partial}{\partial z}\{(\rho_a+\rho_c+\rho_i)w\}\right.$$
$$\left.+\frac{\partial}{\partial z}\{\rho_r(V_r-w)+\rho_s(V_s-w)+\rho_g(V_g-w)\}. \qquad \text{(C1-2-6)}$$

Thus, the continuity equation is replaced by

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}+\frac{\partial\rho w}{\partial z}=\frac{\partial}{\partial z}(\rho_r V_r+\rho_s V_s+\rho_g V_g)$$

$$=\frac{\partial}{\partial z}(\rho_a V_r q_r+\rho_a V_s q_s+\rho_a V_g q_g). \qquad \text{(C1-2-7)}$$

In the model, the reference value of the density is used for the air-density in the right-hand side of (C1-2-7) for simplicity.

C-1-3 Fundamental equations in conformal map projection

In Ikawa and Saito (1991), terms relating to mapping projection were neglected in the basic equations. In the new model, the map factor relating to arbitrary conformal map projections (see in J-1-3) is taken into account in the basic equations. For example, in Polar stereographic projection, the horizontal coordinates in the model (x, y) corresponding to the Earth's surface of the latitude and longitude (φ, λ) are given by

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x_p+ma\cos\varphi\sin\Delta\lambda \\ y_p-ma\cos\varphi\cos\Delta\lambda \end{pmatrix}, \qquad \text{(C1-3-1)}$$

where (x_p, y_p) is the position of the north pole, $\Delta\lambda$ the deflection of longitude from the standard longitude λ_0, and a the radius of the Earth (Fig. J1-3-1). m is the map factor[1] and becomes unity at the standard latitude φ_0,

$$m=\frac{1+\sin\varphi_0}{1+sin\varphi}. \qquad \text{(C-1-3-2)}$$

Distance on the polar projection map (dx, dy) can be derived by differentiating (C1-3-1):

$$\begin{pmatrix} dx \\ dy \end{pmatrix}=ma\begin{pmatrix} -(1+\sin\varphi)^{-1}\cos\varphi d\varphi\cos\varphi\sin\Delta\lambda-\sin\varphi d\varphi\sin\Delta\lambda+\cos\varphi\cos\Delta\lambda d\lambda \\ (1+\sin\varphi)^{-1}\cos\varphi d\varphi\cos\varphi\cos\Delta\lambda+\sin\varphi d\varphi\cos\Delta\lambda+\cos\varphi\sin\Delta\lambda d\lambda \end{pmatrix}$$

$$=ma\begin{pmatrix} -d\varphi\sin\Delta\lambda+\cos\varphi\cos\Delta\lambda d\lambda \\ d\varphi\cos\Delta\lambda+\cos\varphi\sin\Delta\lambda d\lambda \end{pmatrix}=ma\begin{pmatrix} -\sin\Delta\lambda & cos\Delta\lambda \\ \cos\Delta\lambda & sin\Delta\lambda \end{pmatrix}\begin{pmatrix} d\varphi \\ \cos\varphi d\lambda \end{pmatrix}. \qquad \text{(C1-3-3)}$$

Considering the real distance on the Earth's surface, (dx_s, dy_s) is

$$\begin{pmatrix} dx_s \\ dy_s \end{pmatrix}=\begin{pmatrix} a\cos\varphi d\lambda \\ ad\varphi \end{pmatrix}. \qquad \text{(C1-3-4)}$$

The relationship between the differentials is given by the following rotational transformation:

$$\begin{pmatrix} dx \\ dy \end{pmatrix}=m\begin{pmatrix} \cos\Delta\lambda & -\sin\Delta\lambda \\ \sin\Delta\lambda & \cos\Delta\lambda \end{pmatrix}\begin{pmatrix} dx_s \\ dy_s \end{pmatrix}. \qquad \text{(C1-3-5)}$$

Since the velocity is written in the (x, y) coordinate system as

$$u=\frac{1}{m}\frac{dx}{dt}, \qquad \text{(C1-3-6)}$$

$$v=\frac{1}{m}\frac{dy}{dt}, \qquad \text{(C1-3-7)}$$

the relationship between the winds $\{u_s=dx_s/dt,\ v_s=dy_s/dt\}$ in the spherical coordinate system and $\{u, v\}$ is given by

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[1]Strictly speaking, the map factor depends on the height h as

$$m'=\frac{1}{1+h/a}m. \qquad \text{(C1-3-2)'}$$

In this model, we neglect h/a to 1 and substitute m *for* m'.

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\Delta\lambda & -\sin\Delta\lambda \\ \sin\Delta\lambda & \cos\Delta\lambda \end{pmatrix} \begin{pmatrix} u_s \\ v_s \end{pmatrix} \quad \text{(C1-3-8)}$$

Taking the third coordinate z in the upward vertical direction and defining w as $w=dz/dt$, the momentum equations in (x, y, z) are written as follows:

$$\frac{du}{dt} = Cor_1 + Crv_1 - \frac{1}{\rho} m \frac{\partial p}{\partial x} + Dif_1, \quad \text{(C1-3-9)}$$

$$\frac{dv}{dt} = Cor_2 + Crv_2 - \frac{1}{\rho} m \frac{\partial p}{\partial y} + Dif_2, \quad \text{(C1-3-10)}$$

$$\frac{dw}{dt} = Cor_3 + Crv_3 - \frac{1}{\rho} m \frac{\partial p}{\partial z} + Dif_3, \quad \text{(C1-3-11)}$$

where Dif stands for the diffusion terms, and the subscripts 1, 2 and 3 correspond to components x, y and z. Cor and Crv are the following Coriolis and curvature terms[2] (*e.g.,* Kikuchi. 1975).

$$Cor_1 = 2\Omega\sin\varphi v - 2\Omega\cos\varphi\cos\lambda w, \quad \text{(C1-3-12)}$$

$$Cor_2 = -2\Omega\cos\varphi\sin\Delta\lambda w - 2\Omega\sin\varphi u, \quad \text{(C1-3-13)}$$

$$Cor_3 = 2\Omega\cos\varphi\cos\Delta\lambda u + 2\Omega\cos\varphi\sin\Delta\lambda v, \quad \text{(C1-3-14)}$$

$$Crv_1 = m^2 v\{v\frac{\partial}{\partial x}(\frac{1}{m}) - u\frac{\partial}{\partial y}(\frac{1}{m})\} - \frac{uw}{a}, \quad \text{(C1-3-15)}$$

$$Crv_2 = m^2 u\{u\frac{\partial}{\partial y}(\frac{1}{m}) - v\frac{\partial}{\partial x}(\frac{1}{m})\} - \frac{vw}{a}, \quad \text{(C1-3-16)}$$

$$Crv_3 = \frac{u^2+v^2}{a}. \quad \text{(C1-3-17)}$$

With no precipitation, the continuity equation becomes

$$\frac{1}{\rho}\frac{d\rho}{dt} + m^2\{\frac{\partial}{\partial x}(\frac{u}{m}) + \frac{\partial}{\partial y}(\frac{v}{m})\} + \frac{\partial w}{\partial z} + \frac{2w}{a} = 0. \quad \text{(C1-3-18)}$$

For Lambert conformal projection and Mercator projection, map factors are given by (J1-3-7) and (J1-3-13), respectively. For these projections, the form of Eqs. (C1-3-5) - (C1-3-18) is not altered, except that $\Delta\lambda$ becomes $c\Delta\lambda$, where c is given by (J1-3-8) for Lambert projection and zero for Mercator projection.

C-1-4 Fundamental equations in flux form

From (C1-3-6) and (C1-3-7), the total derivative in conformal map projection is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + m(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}) + w\frac{\partial}{\partial z}. \quad \text{(C1-4-1)}$$

Substituting (C1-4-1), the continuity equation (C1-3-18) becomes

$$\frac{\partial\rho}{\partial t} + m(u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y}) + w\frac{\partial\rho}{\partial z} + \rho m^2\{\frac{\partial}{\partial x}(\frac{u}{m}) + \frac{\partial}{\partial y}(\frac{v}{m})\} + \rho\frac{\partial w}{\partial z}$$
$$= \frac{\partial\rho}{\partial t} + m^2\{\frac{\partial}{\partial x}(\frac{\rho u}{m}) + \frac{\partial}{\partial y}(\frac{\rho v}{m})\} + \frac{\partial}{\partial z}(\rho w) = 0. \quad \text{(C1-4-2)}$$

Here, we neglected the last term of (C1-3-18), assuming the vertical scale of the motion is considerably smaller

[2]The last terms in (C1-3-15), (C1-3-16) and (C1-3-17) are proportional to $\frac{1}{m^2}\frac{\partial m'}{\partial z}$, and are absent in the limit of $a\to\infty$. However, these terms should be retained so that the set of basic equations approaches that in the spherical coordinates in the limit of $h/a\to 0$ (see L-3).

than the radius of the Earth. With the fall-out of precipitating substances, the above continuity equation becomes

$$\frac{\partial \rho}{\partial t} + m^2\{\frac{\partial}{\partial x}(\frac{\rho u}{m}) + \frac{\partial}{\partial y}(\frac{\rho v}{m})\} + \frac{\partial}{\partial z}(\rho w) = Prc, \qquad \text{(C1-4-3)}$$

where Prc is the right-hand side of (C1-2-7).

For arbitrary variable ϕ, the total derivative can be written from (C1-4-1) as

$$\frac{\rho}{m^2}\frac{d\phi}{dt} = \frac{\rho}{m^2}\frac{\partial \phi}{\partial t} + \frac{\rho u}{m}\frac{\partial \phi}{\partial x} + \frac{\rho v}{m}\frac{\partial \phi}{\partial y} + \frac{\rho w}{m^2}\frac{\partial \phi}{\partial z}. \qquad \text{(C1-4-4)}$$

From (C1-4-3), we obtain

$$\frac{\phi}{m^2}\frac{\partial \rho}{\partial t} + \phi\{\frac{\partial}{\partial x}(\frac{\rho u}{m}) + \frac{\partial}{\partial y}(\frac{\rho v}{m})\} + \frac{\phi}{m^2}\frac{\partial}{\partial z}(\rho w) - \frac{\phi}{m^2}Prc = 0. \qquad \text{(C1-4-5)}$$

Consequently, the total derivative can be written as

$$\frac{\rho}{m^2}\frac{d\phi}{dt} = \frac{1}{m^2}\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x}(\frac{\rho u\phi}{m}) + \frac{\partial}{\partial y}(\frac{\rho v\phi}{m}) + \frac{1}{m^2}\frac{\partial}{\partial z}(\rho w\phi) - \frac{\phi}{m^2}Prc, \qquad \text{(C1-4-6)}$$

or, assuming (C1-3-2), as

$$\frac{\rho}{m}\frac{d\phi}{dt} = \frac{\partial}{\partial t}(\frac{\rho\phi}{m}) + m\{\frac{\partial}{\partial x}(\frac{\rho v\phi}{m}) + \frac{\partial}{\partial y}(\frac{\rho v\phi}{m})\} + \frac{\partial}{\partial z}(\frac{\rho w\phi}{m}) - \frac{\phi}{m}Prc. \qquad \text{(C1-4-7)}$$

Using (C1-4-7)[3], equations (C1-3-9) to (C1-3-11) become

$$\frac{\partial}{\partial t}(\frac{\rho u}{m}) + Adv.U + \frac{\partial p}{\partial x} = Crv.U + Cor.U + Dif.U, \qquad \text{(C1-4-8)}$$

$$\frac{\partial}{\partial t}(\frac{\rho v}{m}) + Adv.V + \frac{\partial p}{\partial y} = Crv.V + Cor.V + Dif.V, \qquad \text{(C1-4-9)}$$

$$\frac{\partial}{\partial t}(\frac{\rho w}{m}) + Adv.W + \frac{1}{m}(\frac{\partial p}{\partial z} + \rho g) = Crv.W + Cor.W + Dif.W. \qquad \text{(C1-4-10)}$$

Here, $Adv.$ corresponds to the second to last terms on the right-hand side of (C1-4-7), *e.g.*, it is expressed for u as

$$Adv.U = m\{\frac{\partial}{\partial x}(\frac{\rho uu}{m}) + \frac{\partial}{\partial y}(\frac{\rho vu}{m})\} + \frac{\partial}{\partial z}(\frac{\rho wu}{m}) - \frac{u}{m}Prc. \qquad \text{(C1-4-11)}$$

$Cor.U$, $Cor.V$ and $Cor.W$ are Coriolis terms (C1-3-12) to (C1-3-14) multiplied by ρ/m. The curvature terms $Crv.U$, $Crv.V$ and $Crv.W$ are expressed by

$$Crv.U = \frac{\rho v}{m}(u\frac{\partial m}{\partial y} - v\frac{\partial m}{\partial x}) - \frac{\rho u}{m}\frac{w}{a}, \qquad \text{(C1-4-12)}$$

$$Crv.V = \frac{\rho u}{m}(u\frac{\partial m}{\partial y} - v\frac{\partial m}{\partial x}) - \frac{\rho v}{m}\frac{w}{a}, \qquad \text{(C1-4-13)}$$

$$Crv.W = \frac{m}{\rho a}\{(\frac{\rho u}{m})^2 + (\frac{\rho v}{m})^2\}. \qquad \text{(C1-4-14)}$$

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[3]Ikawa and Saito's (1991) nonhydrostatic model used the following relation.

$$\bar{\rho}\frac{\partial \phi}{dt} = \frac{\partial \bar{\rho}\phi}{\partial t} + \frac{\partial \bar{\rho}u\phi}{\partial x} + \frac{\partial \bar{\rho}v\phi}{\partial y} + \frac{\partial \bar{\rho}w\phi}{\partial z}. \qquad \text{(C1-4-7')}$$

The above relationship is correct only for anelastic equations but yields errors for quasi-compressible elastic equations. Ikawa (1988) suggested that the errors caused practically no trouble if the sound wave mode was sufficiently damped. However, they may cause computational instability in some cases when a longer time step is used (Saito, 1994b).

The thermodynamic equation is

$$\frac{d\theta}{dt}=\frac{\partial\theta}{\partial t}+Adv.\theta=\frac{Q}{C_p\pi}+Dif.\theta. \tag{C1-4-15}$$

where the advection term in flux form is given as

$$Adv.\theta=[m\{\frac{\partial}{\partial x}(\frac{\rho u\theta}{m})+\frac{\partial}{\partial y}(\frac{\rho v\theta}{m})\}+\frac{\partial}{\partial z}(\frac{\rho w\theta}{m})-\frac{\theta}{m}(Prc-\underline{\frac{\partial\rho}{\partial t}})]\frac{m}{\rho}. \tag{C1-4-16}$$

The underlined term is the divergence (sum of the second to fourth terms on the left-hand side of (C1-4-3)).

C-2. Fundamental equations in terrain-following coordinates

C-2-1 Equations in terrain-following coordinates

Following Gal-Chen and Somerville (1975) and Clark (1977), we introduce the terrain-following vertical coordinate

$$z^*=\frac{H(z-z_s)}{H-z_s}, \tag{C2-1-1}$$

and components of the metric tensor for the coordinate transformations:

$$G^{\frac{1}{2}}=1-\frac{z_s}{H}, \tag{C2-1-2}$$

$$G^{\frac{1}{2}}G^{13}=(\frac{z^*}{H}-1)\frac{\partial z_s}{\partial x}, \tag{C2-1-3}$$

$$G^{\frac{1}{2}}G^{23}=(\frac{z^*}{H}-1)\frac{\partial z_s}{\partial y}. \tag{C2-1-4}$$

Here, z_s is the surface height and H is the model top height. Applying the chain rule for the coordinate transformation from (x, y, z) to (x, y, z^*), the following relations are obtained for any arbitrary variable ϕ:

$$G^{\frac{1}{2}}\frac{\partial\phi}{\partial x}=\frac{\partial}{\partial x}(G^{\frac{1}{2}}\phi)+\frac{\partial}{\partial z^*}(G^{\frac{1}{2}}G^{13}\phi), \tag{C2-1-5}$$

$$G^{\frac{1}{2}}\frac{\partial\phi}{\partial y}=\frac{\partial}{\partial y}(G^{\frac{1}{2}}\phi)+\frac{\partial}{\partial z^*}(G^{\frac{1}{2}}G^{23}\phi), \tag{C2-1-6}$$

$$G^{\frac{1}{2}}\frac{\partial\phi}{\partial z}=\frac{\partial\phi}{\partial z^*}. \tag{C2-1-7}$$

The cotinuity equation (C1-4-3) is rewritten as follows:

$$G^{\frac{1}{2}}\frac{\partial\rho}{\partial t}+DIVT(U,V,W)=PRC, \tag{C2-1-8}$$

where U, V, and W are wind components multiplied by $\rho G^{1/2}/m$. These are taken as the prognostic variables:

$$U=\frac{\rho G^{\frac{1}{2}}u}{m}, \tag{C2-1-9}$$

$$V=\frac{\rho G^{\frac{1}{2}}v}{m}, \tag{C2-1-10}$$

$$W=\frac{\rho G^{\frac{1}{2}}w}{m}, \tag{C2-1-11}$$

and $DIVT$ is the total divergence in z^* coordinate calculated by

$$DIVT(U,V,W)=m^2(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y})+m\frac{\partial W^*}{\partial z^*}. \tag{C2-1-12}$$

W^* is the vertical momentum in z^* coordinate defined by

$$W^* = \frac{\rho G^{\frac{1}{2}}}{m}\frac{dz^*}{dt} = \frac{1}{G^{\frac{1}{2}}}\{W + m(G^{\frac{1}{2}}G^{13}U + G^{\frac{1}{2}}G^{23}V)\}, \qquad \text{(C2-1-13)}$$

and PRC is the divergence of the fall-out of water substances written in z^* coordinate :

$$PRC = \frac{\partial}{\partial z^*}(\rho_a V_r q_r + \rho_a V_s q_s + \rho_a V_g q_g). \qquad \text{(C2-1-14)}$$

Momentum equations (C1-4-8) to (C1-4-10) are rewritten as follows :

$$\frac{\partial U}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial G^{\frac{1}{2}}G^{13}P}{G^{\frac{1}{2}}\partial z^*} = -ADVU + RU, \qquad \text{(C2-1-15)}$$

$$\frac{\partial V}{\partial t} + \frac{\partial P}{\partial y} + \frac{\partial G^{\frac{1}{2}}G^{13}P}{G^{\frac{1}{2}}\partial z^*} = -ADVU + RV, \qquad \text{(C2-1-16)}$$

$$\frac{\partial W}{\partial t} + \frac{1}{mG^{\frac{1}{2}}}\frac{\partial P}{\partial z^*} = \frac{1}{m}BUOY - ADVW + RW, \qquad \text{(C2-1-17)}$$

where P denotes $(p\text{-}\bar{p})G^{1/2}$. $BUOY$ is the buoyancy term, which is defined by the deviation of the density as

$$BUOY = -(\rho - \bar{\rho})gG^{\frac{1}{2}}. \qquad \text{(C2-1-18)}$$

Here, overbars indicate the variables of the reference atmosphere in which we assume hydrostatic balance. $ADVU$, $ADVV$ and $ADVW$ are the advection terms for U, V and W. For example, $ADVU$ is given as

$$ADVU = m\{\frac{\partial Uu}{\partial x} + \frac{\partial Vu}{\partial y}\} + \frac{\partial W^*u}{\partial z^*} - \frac{u}{m}PRC. \qquad \text{(C2-1-19)}$$

RU, RV and RW represent the residual terms including the curbature, Coriolis and diffusion terms :

$$RU = f_3 V - f_2 W + V(u\frac{\partial m}{\partial y} - v\frac{\partial m}{\partial x}) - U\frac{w}{a} + DIF.U, \qquad \text{(C2-1-20)}$$

$$RV = f_1 W - f_3 U - U(u\frac{\partial m}{\partial y} - v\frac{\partial m}{\partial x}) - V\frac{w}{a} + DIF.V, \qquad \text{(C2-1-21)}$$

$$RW = f_2 U - f_1 V + \frac{m}{\rho G^{\frac{1}{2}}}\frac{(U^2 + V^2)}{a} + DIF.W. \qquad \text{(C2-1-22)}$$

The pressure prognostic equation is obtained from (C1-2-4) and (C2-1-8) as

$$\frac{\partial P}{\partial t} + C_m{}^2(-PFT + DIVT - PRC) = dif.P, \qquad \text{(C2-1-23)}$$

where $dif.P$ is an additional term that comes from Rayleigh damping in pressure, and C_m is

$$C_m{}^2 = \frac{C_p}{C_v}R\theta_m(\frac{p}{p_0})^{R/C_p}. \qquad \text{(C2-1-24)}$$

If there are no precipitable water substances, C_m is reduced to the speed of sound waves given by (C2-2-8). PFT represents the thermal expansion of air[1], which is expressed as

$$PFT = \frac{\rho G^{\frac{1}{2}}}{\theta_m}\frac{\partial \theta_m}{\partial t}. \qquad \text{(C2-1-25)}$$

[1]Some nonhydrostatic models (e.g., Klemp and Wilhemson, 1978 ; Pielke et al., 1992 ; Dudhia, 1993) omit this term from the pressure equations to save computation time and to avoid numerical problems. However, this term represents a substantial part of the state equation and is important to evaluate density perturbation (=buoyancy in our model) accurately. As Doms and Schaettler (1997) mentioned, omitting this term may cause significant problems in numerical weather prediction and the associated data assimilation cycle.

There is no change in the thermodynamic equation.

$$\frac{d\theta}{dt}=\frac{\partial\theta}{\partial t}+ADV.\theta=\frac{Q}{C_p\pi}+Dif.\theta, \qquad \text{(C2-1-26)}$$

except the following modification of the advection term;

$$ADV.\theta=\{m(\frac{\partial U\theta}{\partial x}+\frac{\partial V\theta}{\partial y})+\frac{\partial W^*\theta}{\partial z^*}\frac{\theta}{m}(PRC-G^{\frac{1}{2}}\frac{\partial\rho}{\partial t})\}\frac{m}{\rho G^{\frac{1}{2}}}$$

$$=\{m^2(\frac{\partial U\theta}{\partial x}+\frac{\partial V\theta}{\partial y})+m\frac{\partial W^*\theta}{\partial z^*}-\theta DIVT(U,V,W)\}/\rho G^{\frac{1}{2}}. \qquad \text{(C2-1-27)}$$

C-2-2 Quasi-compressible approximation

A quasi-compressible approximated version is historically developed as an elastic version of Ikawa and Saito (1991)'s nonhydrostatic model. The field variables $\phi(x, y, z, t)$ are divided into a horizontally uniform reference basic state $\bar{\phi}(z)$ and residual perturbation $\phi'(x, y, z, t)$ as follows:

$$\phi(x,y,z,t)=\bar{\phi}(z)+\phi'(x,y,z,t). \qquad \text{(C2-2-1)}$$

Note that $\phi(z)$ depends on not only z^* but also on x and y, in the terrain-following coordinate system; $\bar{\phi}(z)=\phi(x, y, z^*)$. The set of quasi-compressible equations is obtained by setting $m=1$, $PRC=0$, and $\rho(x, y, z, t)=\bar{\rho}(z)$ in (C2-1-9) to (C2-1-27) except for $BUOY$ in (C2-1-17):

$$G^{\frac{1}{2}}\frac{\partial\rho}{\partial t}+DIVT(U,V,W)=0, \qquad \text{(C2-2-2)}$$

$$U=\bar{\rho}G^{\frac{1}{2}}u,$$

$$V=\bar{\rho}G^{\frac{1}{2}}v, \qquad \text{(C2-2-3)}$$

$$W=\bar{\rho}G^{\frac{1}{2}}w,$$

$$\frac{\partial U}{\partial t}+\frac{\partial P}{\partial x}+\frac{\partial G^{\frac{1}{2}}G^{13}P}{G^{\frac{1}{2}}\partial z^*}=ADVU+RU, \qquad \text{(C2-2-4)}$$

$$\frac{\partial V}{\partial t}+\frac{\partial P}{\partial y}+\frac{\partial G^{\frac{1}{2}}G^{23}P}{G^{\frac{1}{2}}\partial z^*}=ADVV+RU. \qquad \text{(C2-2-5)}$$

Hire, $ADVU$, $ADVV$, and $ADVW$ are defined as[2]

$$ADVU=\frac{\partial Uu}{\partial x}+\frac{\partial Vu}{\partial y}+\frac{\partial W^*u}{\partial z^*}. \qquad \text{(C2-2-6)}$$

A density perturbation in the vertical equation is divided into two components by the following approximations:

$$-\rho' g\cong\{\bar{\rho}(\frac{\theta_v'}{\bar{\theta}_v}-q_c-q_r-q_s-q_g-q_i)-\frac{p'}{C_s^2}\}g, \qquad \text{(C2-2-7)}$$

where

$$C_s{}^2=\frac{C_p}{C_v}R\bar{\theta}_v(\frac{p}{p_0})^{R/C_p}, \qquad \text{(C2-2-8)}$$

Using the above, the vertical momentum equation becomes

$$\frac{\partial W}{\partial t}+\frac{1}{G^{\frac{1}{2}}}\frac{\partial P}{\partial z^*}+\frac{P}{C_s{}^2}g=BUOY'-ADVW+RW, \qquad \text{(C2-2-9)}$$

where the buoyancy term is given by

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[2]As mentioned in C-1-4, this form requires the anelastic relation (C2-3-1) and is not exact for (C2-2-2).

$$BUOY' \equiv \bar{\rho}G^{\frac{1}{2}}(\frac{\theta_v'}{\bar{\theta}_v} - q_c - q_r - q_s - q_g - q_i)g. \qquad \text{(C2-2-10)}$$

The pressure equation is as follows

$$\frac{\partial P}{\partial t} + C_s^2(-PFT + DIVT) = dif.P, \qquad \text{(C2-2-11)}$$

where

$$PFT = \frac{1}{g}\frac{\partial BUOY'}{\partial t}. \qquad \text{(C2-2-12)}$$

C-2-3 Anelastic approximation

In the anelastic system, the first term of the continuity equation (C2-2-2) is neglected in order to remove the sound waves following Ogura and Phillips (1962),

$$DIVT(U, V, W) = 0, \qquad \text{(C2-3-1)}$$

where $m=1$ is set in (C2-1-12) and the three components of the momentum are defined by (C2-2-3) as in the quasi-compressible model. The linearized momentum equations are the same as in (C2-2-4), (C2-2-5) and (C2-2-9), and the flux form (C2-2-6) is exact for (C2-3-1). Pressure is diagnosed by the anelastic continuity equation and momentum equations (see C-3).

C-2-4 Hydrostatic version of the anelastic model

The hydrostatic version of the anelastic model is obtained by degenerating the vertical momentum equation (C2-2-9) into the hydrostatic approximation as

$$\frac{1}{G^{\frac{1}{2}}}\frac{\partial P}{\partial z^*} + \frac{P}{C_s^2}g = BUOY'. \qquad \text{(C2-4-1)}$$

Vertical momentum W is determined by the continuity equation (C2-3-1). The method for calculating the pressure is presented in subsection C-3-4.

C-3. Pressure equations

Since elastic nonhydrostatic models include sound waves in their solutions, the maximum time step is restricted by the speed of sound waves if a simple leap-frog time integration scheme is used. To overcome this problem, current nonhydrostatic models treat sound waves in two schemes: one that treats sound waves implicitly in the vertical direction and explicitly in the horizontal (HE-VI scheme; *e.g.,* Klemp and Wilhelmson, 1978) and another that treats sound waves implicitly in both the horizontal and vertical directions (HI-VI scheme; *e.g.,* Tapp and White, 1976). Generally, a time-splitting scheme whereby high-frequency terms are evaluated at a shorter time step level is used in the HE-VI scheme.

The explicit treatment of sound waves in the horizontal directions presumes that the horizontal resolution is much finer than the vertical resolution. Consequently, the HE-VI scheme may need a very short time step when the horizontal grid interval becomes as small as the vertical grid interval. On the other hand, the time step in the HI-VI scheme is not restricted by the sound wave speed, but it is necessary to solve a three-dimensional Helmholtz equation for pressure. This characteristic feature of the HI-VI scheme may become a disadvantage in massive computation using a distributed memory parallel computer.

From the above point of view, some recent nonhydrostatic models tend to prepare two options to treat sound waves. For example, an HI-VI version of Lokal-modell (Doms and Schaettler, 1997) of the Deutscher Wetterdienst was recently developed by Thomas et al. (1999), and an HI-VI option is being developed in the WRF model project at NCEP. In JMA, a new HE-VI scheme was incorporated into the MRI nonhydrostatic model as the first step to develop the MRI/NPD unified nonhydrostatic model.

Most operational hydrostatic models, as well as some nonhydrostatic models (*e.g.,* Tanguay *et al.,* 1990; Golding, 1992), treat gravity waves implicitly to maintain computational stability and efficiency. Implicit treatment of the gravity waves is one of our future subjects, though it may deform not only the high-frequency gravity waves but also the low-frequency gravity waves.

C-3-1 Pressure tendency equation in HI-VI scheme

In this model, we treat implicitly only the sound waves following the E-HI-VI scheme described in Ikawa (1988) and Ikawa and Saito (1991) for their quasi-compressible model. Before presenting the formulation, we redefine the buoyancy term $BUOY$ in this section in order to unify the two expressions of buoyancy (C2-1-18) and (C2-2-10). Introducing a switching parameter σ, which takes zero for direct computation of the buoyancy from density perturbation and unity for conventional computation by the temperature perturbation, the term can be rewritten as

$$BUOY \equiv \sigma \frac{\rho G^{\frac{1}{2}} \theta_m'}{\theta_m} g + (1-\sigma)(\bar{\rho}-\rho) g G^{\frac{1}{2}}. \qquad \text{(C3-1-1)}$$

Usually, σ is zero for the fully compressible mode and unity for quasi-compressible approximation (in this case, the first term on the right-hand side becomes (C2-2-10). Using σ, vertical momentum equations can be expressed as

$$\frac{\partial W}{\partial t} + \frac{1}{mG^{\frac{1}{2}}}\frac{\partial P}{\partial z^*} + \sigma \frac{P}{mC_m{}^2} g = \frac{1}{m} BUOY - ADVW + RW. \qquad \text{(C3-1-2)}$$

In the E-HI-VI scheme, momentum equations (C2-1-15), (C2-1-16) and (C3-1-2) are represented in finite difference form as follows:

$$\frac{U^{it+1}-U^{it-1}}{2\Delta t} + \frac{\partial \bar{P}^t}{\partial x} + -(ADVU - RU + \frac{\partial G^{\frac{1}{2}} G^{13} P}{G^{\frac{1}{2}} \partial z^*}), \qquad \text{(C3-1-3)}$$

$$\frac{V^{it+1}-V^{it-1}}{2\Delta t} + \frac{\partial \bar{P}^t}{\partial y} + -(ADVV - RV + \frac{\partial G^{\frac{1}{2}} G^{23} P}{G^{\frac{1}{2}} \partial z^*}), \qquad \text{(C3-1-4)}$$

$$\frac{W^{it+1}-W^{it-1}}{2\Delta t} + \overline{\overline{\frac{1}{mG^{\frac{1}{2}}}}} \frac{\partial \bar{P}^t}{\partial z^*} + \overline{\overline{\frac{g}{mC_m{}^2}}} \bar{P}^t$$

$$= \frac{1}{m} BUOY - (ADVW - RW) + (\overline{\overline{\frac{1}{mG^{\frac{1}{2}}}}} - \frac{1}{mG^{\frac{1}{2}}}) \frac{\partial P}{\partial z^*} + (\overline{\overline{\frac{g}{mC_m{}^2}}} - \sigma \frac{g}{mC_m{}^2}) P. \qquad \text{(C3-1-5)}$$

Here, terms marked with a double overbar denote averaged quantity on a z^* surface and superscripts $it+1$ and $it-1$ represent the time levels in the leap-frog integration. Terms marked with a single overbar together with superscript t denote the weighted averaged values between $it+1$ and $it-1$ time levels defined by

$$\bar{A}^t \equiv \frac{1+\alpha}{2} A^{it+1} + \frac{1-\alpha}{2} A^{it-1} \equiv \frac{\Delta^2 A}{2} + A^{it}, \qquad \text{(C3-1-6)}$$

where α is the weight parameter, which is currently set to 0.5. The last terms on the left-hand and right-hand sides of (C3-1-5) are pressure perturbation components in the buoyancy term, which must be treated implicitly to maintain computational stability. Thermodynamic equation (C2-1-26) is represented in a simple difference form since the gravity waves are not treated implicitly in our model.

Pressure prognostic equation (C2-1-23) is represented in finite difference form as

$$\frac{\partial P}{\partial t}+\overline{\overline{C_m^2}}DIVS(\overline{U^t},\overline{V^t},\overline{W^t})$$

$$=C_m^2(PFT-DIVT(U,V,W)+PRC)+\overline{\overline{C_m^2}}DIVS(U,V,W)+dif.P, \qquad \text{(C3-1-7)}$$

where $DIVS$ is the linearized separable part of the total divergence, which is defined by

$$DIVS(U,V,W)=\overline{\overline{m^2}}\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)+\overline{\overline{\frac{m}{G^{\frac{1}{2}}}}}\frac{\partial W}{\partial z^*}. \qquad \text{(C3-1-8)}$$

Substituting the following relation

$$\frac{A^{it+1}-A^{it-1}}{2\Delta t}=\frac{\Delta^2 A}{2(1+\alpha)\Delta t}+\frac{A^{it}-A^{it-1}}{(1+\alpha)\Delta t}, \qquad \text{(C3-1-9)}$$

into (C3-1-3) to (C3-1-5) and (C3-1-7), we obtain the following formulas:

$$\frac{\Delta^2 U}{(1+\alpha)\Delta t}+\frac{\partial\Delta^2 P}{\partial x}=-2ADVU', \qquad \text{(C3-1-10)}$$

$$\frac{\Delta^2 V}{(1+\alpha)\Delta t}+\frac{\partial\Delta^2 P}{\partial y}=-2ADVV', \qquad \text{(C3-1-11)}$$

$$\frac{\Delta^2 W}{(1+\alpha)\Delta t}+\overline{\overline{\frac{1}{mG^{\frac{1}{2}}}}}\frac{\partial\Delta^2 P}{\partial z^*}+\frac{\overline{\overline{g}}}{m\overline{\overline{C_m^2}}}\Delta^2 P=-2ADVW', \qquad \text{(C3-1-12)}$$

$$\frac{\Delta^2 P}{(1+\alpha)\Delta t}+\overline{\overline{C_m^2}}DIVS(\Delta^2 U,\Delta^2 V,\Delta^2 W)=-\overline{\overline{C_m^2}}2ADVP', \qquad \text{(C3-1-13)}$$

where the right-hand sides are the modified advection terms:

$$ADVU'=ADVU-RU+\frac{U^{it}-U^{it-1}}{(1+\alpha)\Delta t}+\frac{\partial P^{it}}{\partial x}+\frac{\partial G^{\frac{1}{2}}G^{13}P^{it}}{G^{\frac{1}{2}}\partial z^*}, \qquad \text{(C3-1-14)}$$

$$ADVV'=ADVV-RV+\frac{V^{it}-V^{it-1}}{(1+\alpha)\Delta t}+\frac{\partial P^{it}}{\partial y}+\frac{\partial G^{\frac{1}{2}}G^{23}P^{it}}{G^{\frac{1}{2}}\partial z^*}, \qquad \text{(C3-1-15)}$$

$$ADVW'=ADVW-RW-\frac{1}{m}BUOY+\frac{W^{it}-W^{it-1}}{(1+\alpha)\Delta t}+\frac{1}{m}\left(\frac{1}{G^{\frac{1}{2}}}\frac{\partial}{\partial z^*}+\sigma\frac{g}{C_m^2}\right)P^{it}, \qquad \text{(C3-1-16)}$$

$$ADVP'=\frac{1}{\overline{\overline{C_m^2}}}\left\{\frac{P^{it}-P^{it-1}}{(1+\alpha)\Delta t}-C_m^2(PFT-DIVT(U,V,W)+PRC)-dif.P\right\}. \qquad \text{(C3-1-17)}$$

Substitution of (C3-1-10) to (C3-1-12) into (C3-1-13) to eliminate $\Delta^2 U$, $\Delta^2 V$, and $\Delta^2 W$, yields

$$\frac{\Delta^2 P}{\overline{\overline{C_m^2}}(1+\alpha)^2(\Delta t)^2}-DIVS\left(2ADVU'+\frac{\partial\Delta^2 P}{\partial x},2ADVV'+\frac{\partial\Delta^2 P}{\partial y},2ADVW'+\overline{\overline{\frac{1}{mG^{\frac{1}{2}}}}}\frac{\partial\Delta^2 P}{\partial z^*}+\overline{\overline{\frac{g}{mC_m^2}}}\Delta^2 P\right)$$

$$=-\frac{2}{(1+\alpha)\Delta t}ADVP'. \qquad \text{(C3-1-18)}$$

Arranging the above, we obtain the following Helmholtz-type pressure tendency equation:

$$\overline{\overline{m^2}}\left(\frac{\partial^2\Delta^2 P}{\partial x^2}+\frac{\partial^2\Delta^2 P}{\partial y^2}\right)+\overline{\overline{\frac{m}{G^{\frac{1}{2}}}}}\,\overline{\overline{\frac{1}{mG^{\frac{1}{2}}}}}\frac{\partial^2\Delta^2 P}{\partial z^{*2}}+\overline{\overline{\frac{m}{G^{\frac{1}{2}}}}}\frac{\partial}{\partial z^*}\left(\overline{\overline{\frac{g}{mC_m^2}}}\Delta^2 P\right)$$

$$-\frac{\Delta^2 P}{\overline{\overline{C_m^2}}(1+\alpha)^2(\Delta t)^2}=2\left\{\frac{ADVP'}{(1+\alpha)\Delta t}-DIVS(ADVU',ADVV',ADVW')\right\}. \qquad \text{(C3-1-19)}$$

This equation is solved directly by the Dimension Reduction Method discussed in D-3. In Saito (1997), the model assumed

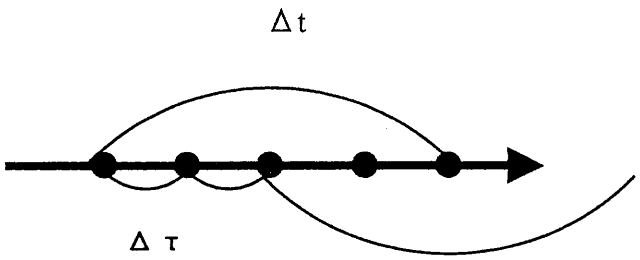
$$\overline{\overline{m^2}}\cong 1,\ \overline{\overline{\frac{m}{G^{\frac{1}{2}}}}}\cong\overline{\overline{\frac{1}{mG^{\frac{1}{2}}}}}\cong\overline{\overline{\frac{1}{G^{\frac{1}{2}}}}},\ \overline{\overline{\frac{g}{mC_m^{\ 2}}}}\cong\overline{\overline{\frac{g}{C_m^{\ 2}}}}, \qquad \text{(C3-1-20)}$$

for simplicity on the understanding that the standard latitude φ_0 in (C1-3-2) is taken at the center of the model domain. However, the above approximation (premise) was removed in the latest version (Saito, 2000).

C-3-2 Formulation of HE-VI scheme

Referring to the E-HE-VI scheme described in Ikawa and Saito (1991), the time-splitting, horizontally explicit version of the MRI mesoscale nonhydrostatic model is presented in the following. The Numerical Prediction Division of JMA incorporated this new scheme into MRI-NHM as a joint program with MRI to develop the MRI/NPD unified mesoscale model (Muroi et al., 1999) (see L-2).

In the E-HE-VI scheme, the time integration procedure is divided into two parts to avoid severe restriction of time interval for integration. Terms related to the sound mode are treated explicitly in the horizontal direction and implicitly in the vertical direction. These terms are integrated with a short time step ($\Delta\tau$). The other terms that include the advection term, friction term and physical processes are integrated with long time step (Δt). In the short time step integration, the horizontal velocity is integrated first with a forward scheme. The vertical velocity and pressure field are then solved with a backward scheme. In the long time step integration, a three-time level scheme with a time filter is applied.



Forward time integration of (C2-1-15) and (C2-1-16) and backward integration of (C3-1-2) and (C2-1-23) are represented in finite difference form as

$$\frac{U^{\tau+\Delta t}-U^{\tau}}{\Delta\tau}+\frac{\partial P^{\tau}}{\partial x}+\frac{\partial G^{\frac{1}{2}}G^{13}P^{\tau}}{G^{\frac{1}{2}}\partial z^*}=-(ADVU+RU), \qquad \text{(C3-2-1)}$$

$$\frac{V^{\tau+\Delta t}-V^{\tau}}{\Delta\tau}+\frac{\partial P^{\tau}}{\partial y}+\frac{\partial G^{\frac{1}{2}}G^{23}P^{\tau}}{G^{\frac{1}{2}}\partial z^*}=-(ADVV+RV), \qquad \text{(C3-2-2)}$$

$$\frac{W^{\tau+\Delta\tau}-W^{\tau}}{\Delta\tau}+\frac{1}{mG^{\frac{1}{2}}}\frac{\partial P^{\beta}}{\partial z^*}+\frac{g}{mC_m^{\ 2}}P^{\beta}=\frac{1}{m}BUOY-(ADVW-RW)+(1-\sigma)\frac{g}{mC_m^{\ 2}}P, \qquad \text{(C3-2-3)}$$

$$\frac{P^{\tau+\Delta t}-P^{\tau}}{\Delta\tau}+C_m^{\ 2}(-PFT+m^2(\frac{\partial U^{\gamma}}{\partial x}+\frac{\partial V^{\gamma}}{\partial y})$$

$$+m\frac{\partial}{\partial z^*}[\frac{1}{G^{\frac{1}{2}}}\{W^{\beta}+m(G^{\frac{1}{2}}G^{13}U^{\gamma}+G^{\frac{1}{2}}G^{23}V^{\gamma})\}]-PRC)=dif.P, \qquad \text{(C3-2-4)}$$

where

$$U^{\gamma}=\frac{1+\gamma}{2}U^{\tau+\Delta\tau}+\frac{1-\gamma}{2}U^{\tau}, \qquad \text{(C3-2-5)}$$

$$V^{\gamma}+\frac{1+\gamma}{2}V^{\tau+\Delta\tau}+\frac{1-\gamma}{2}V^{\tau}, \qquad \text{(C3-2-6)}$$

$$W^{\beta}=\frac{1+\beta}{2}W^{\tau+\Delta\tau}+\frac{1-\beta}{2}W^{\tau}$$

$$=\frac{\Delta\tau(1+\beta)}{2}\left(\frac{W^{\tau+\Delta\tau}-W^{\tau}}{\Delta\tau}\right)+W^{\tau}=\frac{\Delta\tau(1+\beta)}{2}\delta W+W^{\tau}. \qquad \text{(C3-2-7)}$$

Here, the left-hand sides are calculated in the short time step integration, and the right-hand sides are evaluated in the long time step integration. P^{β} is similar to (C3-2-7). $\sigma=1$, $\beta=0.5$ and $\gamma=0$ are used in the current version of the model.

Substituting the above equations into (C3-2-4), we obtain

$$\frac{2}{\Delta\tau(1+\beta)}(P^{\beta}-P^{\tau})+C_m^2((-PFT+m^2DIVH(U^{\gamma},V^{\gamma})$$

$$+\underline{m\frac{\partial}{\partial z^*}[\frac{1}{G^{\frac{1}{2}}}\{\frac{\Delta\tau(1+\beta)}{2}}\delta W+W^{\tau}\}]-PRC))=dif.P, \qquad \text{(C3-2-8)}$$

where $DIVH$ stands for horizontal divergence

$$DIVH(U,V)\equiv\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial}{\partial z^*}(G^{13}U+G^{23}V). \qquad \text{(C3-2-9)}$$

The under-lined part of (C3-2-8) becomes

$$-(\frac{1}{G^{\frac{1}{2}}})^2\frac{\Delta\tau(1+\beta)}{2}\frac{\partial^2P^{\beta}}{\partial z^*}-\frac{1}{G^{\frac{1}{2}}}\frac{\Delta\tau(1+\beta)}{2}\frac{\partial}{\partial z^*}(\frac{g}{C_m^2}P^{\beta})$$

$$+\frac{m}{G^{\frac{1}{2}}}\frac{\Delta\tau(1+\beta)}{2}\frac{\partial}{\partial z^*}\{r.h.s.\,(\text{C3-2-3})\}. \qquad \text{(C3-2-10)}$$

Finally, we obtain the following one-dimensional Helmholtz-type pressure equation

$$\frac{\partial^2P^{\beta}}{\partial z^{*2}}+G^{\frac{1}{2}}\frac{\partial}{\partial z^*}(\frac{g}{C_m^2}P^{\beta})-(G^{\frac{1}{2}})^2\{\frac{2}{C_m\Delta\tau(1+\beta)}\}^2P^{\beta}=FP.HE.INV+FP.HE.VAR, \qquad \text{(C3-2-11)}$$

where $FP.HE.INV$ is the invariable term during short time step integration in the forcing terms

$$PP.HE.INV=\frac{2(G^{\frac{1}{2}})^2}{\Delta\tau(1+\beta)}(PFT+PRC+\frac{dif.P}{C_m^2})$$

$$+G^{\frac{1}{2}}\frac{\partial}{\partial z^*}\{BUOY-m(ADVW-RW)+(1-\sigma)\frac{g}{C_m^2}P\}, \qquad \text{(C3-2-12)}$$

and $FP.HE.VAR$ is the variable term during short time step integration in the forcing terms

$$FP.HE.VAR=\frac{2(G^{\frac{1}{2}})^2}{\Delta\tau(1+\beta)}\{m^2DIVH(U^{\gamma},V^{\gamma})+\frac{m}{G^{\frac{1}{2}}}\frac{\partial W^{\tau}}{\partial z^*}\}-\{\frac{2G^{\frac{1}{2}}}{C_m\Delta\tau(1+\beta)}\}^2P^{\tau}. \qquad \text{(C3-2-13)}$$

The upper boundary condition may be obtained from $\delta W=0$ as

$$(\frac{1}{mG^{\frac{1}{2}}}\frac{\partial}{\partial z^*}+\frac{g}{mC_m^2})P^{\beta}=-(ADVW-RW)+\frac{1}{m}\{BUOY+(1-\sigma)\frac{g}{C_m^2}P\}. \qquad \text{(C3-2-14)}$$

The lower boundary condition is the same as in (C3-2-14), while in free-slip case, the right-hand side requires the following extra term

$$-\delta W=m(G^{\frac{1}{2}}G^{13}\delta U+G^{\frac{1}{2}}G^{23}\delta V)=m(G^{\frac{1}{2}}G^{13}\frac{U^{\tau+\Delta t}-U^{\tau}}{\Delta\tau}+G^{\frac{1}{2}}G^{23}\frac{V^{\tau+\Delta t}-V^{\tau}}{\Delta\tau})$$

$$=\frac{1}{\Delta\tau}\{W^{\tau}+m(G^{\frac{1}{2}}G^{13}U^{\tau+\Delta\tau}+G^{\frac{1}{2}}G^{23}V^{\tau+\Delta\tau}),\qquad\text{(C3-2-15)}$$

in order to satisfy the kinematic condition

$$\delta W^{*}=\delta W+m(G^{\frac{1}{2}}G^{13}\delta U+G^{\frac{1}{2}}G^{23}\delta V)=0,$$

$$W^{*\tau}=W^{\tau}+m(G^{\frac{1}{2}}G^{13}U^{\tau}+G^{\frac{1}{2}}G^{23}V^{\tau})=0.\qquad\text{(C3-2-16)}$$

C-3-3 Pressure diagnostic equation in anelastic version

Pressure in the anelastic system is described in Ikawa and Saito (1991). Here, we present it for discussion in later chapters. Substituting (C2-2-4), (C2-2-5) and (C2-2-9) into (C2-3-1) to eliminate time tendencies of U, V and W, we obtain

$$-\frac{\partial DIVT(U,V,W)}{\partial t}=DIVT(\frac{\partial P}{\partial x}+\frac{\partial G^{\frac{1}{2}}G^{13}P}{G^{\frac{1}{2}}\partial z^{*}}+ADVU-RU,$$

$$\frac{\partial P}{\partial y}+\frac{\partial G^{\frac{1}{2}}G^{23}P}{G^{\frac{1}{2}}\partial z^{*}}+ADVV-RV,\frac{1}{G^{\frac{1}{2}}}\frac{\partial P}{\partial z^{*}}+\frac{P}{C_m^{2}}g+ADVW-RW-BUOY'),\qquad\text{(C3-3-1)}$$

where we assume $m=1$ for the anelastic model. Introducing a residual part of the total divergence to the separable part

$$DIVR(U,V,W)=DIVT(U,V,W)-DIVS(U,V,W)$$

$$=\frac{\partial}{\partial z^{*}}\frac{1}{G^{\frac{1}{2}}}\{W+G^{\frac{1}{2}}G^{13}U+G^{\frac{1}{2}}G^{23}V\}-\overline{\overline{\frac{1}{G^{\frac{1}{2}}}}}\frac{\partial W}{\partial z^{*}}$$

$$=\frac{1}{G^{\frac{1}{2}}}\frac{\partial}{\partial z^{*}}(G^{\frac{1}{2}}G^{13}U+G^{\frac{1}{2}}G^{23}V)+(\frac{1}{G^{\frac{1}{2}}}-\overline{\overline{\frac{1}{G^{\frac{1}{2}}}}})\frac{\partial W}{\partial z^{*}},\qquad\text{(C3-3-2)}$$

we obtain the following Poisson-type pressure diagnostic equation:

$$DIVS(\frac{\partial P}{\partial x},\frac{\partial P}{\partial y},\overline{\overline{\frac{1}{G^{\frac{1}{2}}}}}\frac{\partial P}{\partial z^{*}}+\overline{\overline{\frac{g}{C_s^{2}}}}P)+DIVR(\frac{\partial P}{\partial x},\frac{\partial P}{\partial y},\overline{\overline{\frac{1}{G^{\frac{1}{2}}}}}\frac{\partial P}{\partial z^{*}}+\overline{\overline{\frac{g}{C_s^{2}}}}P)$$

$$+DIVT\{\frac{\partial G^{\frac{1}{2}}G^{13}P}{G^{\frac{1}{2}}\partial z^{*}},\frac{\partial G^{\frac{1}{2}}G^{23}P}{G^{\frac{1}{2}}\partial z^{*}},(\frac{1}{G^{\frac{1}{2}}}\frac{\partial}{\partial z^{*}}+\frac{g}{C_s^{2}}-\overline{\overline{\frac{1}{G^{\frac{1}{2}}}}}\frac{\partial}{\partial z^{*}}-\overline{\overline{\frac{g}{C_s^{2}}}})P\}$$

$$+DIVT(ADVU-RU,ADVV-RV,ADVW-RW-BUOY')$$

$$=\frac{1}{2\Delta t}DIVT(U^{\tau-1},V^{\tau-1},W^{\tau-1}),\qquad\text{(C3-3-3)}$$

The right-hand side of the above equation represents the residual divergence computed at the former time level, which is theoretically zero in the anelastic system but non-zero numerically due to computation errors. This term is important for satisfying the continuity equation at the next time level and stabilizing the computation (Clark, 1977; Ikawa and Saito, 1991).

C-3-4 Pressure equation in anelastic hydrostatic version

The hydrostatic model is a degenerate version of the non-hydrostatic model developed by Ikawa and Saito (1991). Clark and Hall (1991) also made a hydrostatic version of their non-hydrostatic model by using almost the same method as used in this subsection. Since the vertical momentum flux W is a diagnostic variable, W is determined by the anelastic continuity equation. The pressure is calculated as follows.

Combining (C2-2-4), (C2-2-5), and (C2-4-1) leads to a diagnostic equation for p' as

$$\frac{\partial^2 p'}{\partial x^2}+\frac{\partial^2 (G^{13}p')}{\partial x \partial z^*}+\frac{\partial^2 p'}{\partial y^2}+\frac{\partial^2 (G^{23}p')}{\partial y \partial z^*}=-\frac{\partial Fx}{\partial x}-\frac{\partial Fy}{\partial y}+\frac{1}{2\Delta t}\frac{\partial (W^{m+1}-W^{m-1})}{\partial z}, \quad \text{(C3-4-1)}$$

where Δt is the time step interval, and the superscript 'm' denotes the value at the time step 'm'. From (C2-4-1), p'_k at the k-th vertical level can be expressed with the pressure at the upper boundary p'_{nz} as

$$p'_k = a_k p_{nz} + b_k, \quad \text{(C3-4-2)}$$

where

$$a_k = \sum_{k=k}^{nz} \frac{G^{\frac{1}{2}} - \Delta z_{k+\frac{1}{2}} \frac{g}{C_s^2}}{G^{\frac{1}{2}} + \Delta z_{k+\frac{1}{2}} \frac{g}{C_s^2}} \quad \text{(C3-4-3)}$$

and

$$b_k = -\sum_{k=k}^{nz} \frac{\Delta z_{k+\frac{1}{2}} BOUY'_{k+\frac{1}{2}}}{G^{\frac{1}{2}} + \Delta z_{k+\frac{1}{2}} \frac{g}{C_s^2}}. \quad \text{(C3-4-4)}$$

Summing (C3-4-1) in the vertical column results in the following diagnostic column pressure equation:

$$\frac{\partial^2}{\partial x^2}\sum_k \Delta z_k p'_k + \frac{\partial}{\partial x}(G^{13}p'_{nz} - G^{13}p'_1) + \frac{\partial^2}{\partial y^2}\sum_k \Delta z_k p'_k + \frac{\partial}{\partial y}(G^{23}p'_{nz} - G^{23}p'_1)$$
$$= -\sum_k \Delta z_k \left(\frac{\partial Fx}{\partial x} + \frac{\partial Fy}{\partial y}\right)_k + \frac{\{(W^{m+1} - W^{m-1})_{nz} - (W^{m+1} - W^{m-1})_1\}}{2\Delta t}, \quad \text{(C3-4-5)}$$

where Δz_k is the k-th vertical grid interval and the subscripts 1 and nz denote the values at the lower and upper boundaries, respectively. The second term on the right-hand side of (C3-4-5) should theoretically be zero because of the lower and upper boundary conditions. However, since W^{m-1}_{nz} calculated from (C2-3-1) becomes non-zero due to round-off errors in numerical simulations, this term remains in (C3-4-5) to guarantee $W^{m+1}_{nz}=0$. Substitution of (C3-4-2) into (C3-4-5) results in the horizontal elliptic equation for p'_{nz}, which can be solved by the same method as in the non-hydrostatic model (see subsection D-3-3).