6 Transport process of moisture and ozone*

In this chapter the transport process of passive quantities, namely, moistute and ozone, is described. Moisture and ozone are passive quantities in the sense that they are conveyed by the advection process besides sources and sinks and do not affect the dynamical fields directly. Therefore their finite differencing schemes seem to be relatively straightforward. There is, however, the difficulty which comes from their large spacial variations and existence of the lower limit (i.e. zero). As for moisture, another difficulty named as CICK ("conditional instability of computational kind") must be avoided. These problems are partially solved by choosing interpolation values for the half-integer levels appropriately as described in this chapter. See Arakawa and Mintz (1974, hereafter referred to as AM) for details.

6.1 Vertical differencing of the moisture equation

The continuity equation for water vapor (0.28) is repeated here with S replaced by $-C$,

$$\frac{\partial}{\partial t}(\pi q) = -\nabla \cdot (\pi \mathbf{v} q) - \frac{\delta(\pi\dot{\sigma}q)}{\delta\sigma} - \pi C \tag{6.1}$$

where C is the sink of water vapor per unit mass of dry air. Using the continuity equation (1. 3), (6.1) may be rewritten as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) q_k = -\frac{1}{(\pi\delta\sigma)_k}\left[(\pi\dot{\sigma})_{k+1/2}(\hat{q}_{k+1/2} - q_k) + (\pi\dot{\sigma})_{k-1/2}(q_k - \hat{q}_{k-1/2})\right] - C_k \tag{6.2}$$

This form of vertical differencing is also used for ozone transport. Consider, first, a moist adiabatic process. Let the saturation mixing ratio be $q^*_k = q^*(T_k, p_k)$. When the layer k is saturated and remains saturated, (6.1) may be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) q^*_k = -\frac{1}{(\pi\delta\sigma)_k}\left[(\pi\dot{\sigma})_{k+1/2}(\hat{q}_{k+1/2} - q^*_k) + (\pi\dot{\sigma})_{k-1/2}(q^*_k - \hat{q}_{k-1/2})\right] - C_k \tag{6.3}$$

and then as

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$$\left(\frac{\partial q^*}{\partial T}\right)_{pk}\left(\frac{\partial}{\partial t}+\mathbf{v}_k\cdot\nabla\right)T_k=-\left(\frac{\partial q^*}{\partial p}\right)_{Tk}\sigma_k\left(\frac{\partial}{\partial t}+\mathbf{v}_k\cdot\nabla\right)\pi_k$$

$$-\frac{1}{(\pi\delta\sigma)_k}\left[(\pi\dot{\sigma})_{k+1/2}(\hat{q}_{k+1/2}-q^*_k)+(\pi\dot{\sigma})_{k-1/2}(q^*_k-\hat{q}_{k-1/2})\right]-C_k \tag{6.4}$$

where

$$\left(\frac{\partial q^*}{\partial T}\right)_{pk}=\left(\frac{\partial q_k^*}{\partial T_k}\right)_{pk},\qquad \left(\frac{\partial q^*}{\partial p}\right)_{Tk}=\left(\frac{\partial q_k^*}{\partial p_k}\right)_{Tk} \tag{6.5}$$

The thermodynamic energy equation is, from (1.11),

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_k\cdot\nabla\right)T_k=-\frac{1}{c_p}\alpha_k\sigma_k\left(\frac{\partial}{\partial t}+\mathbf{v}_k\cdot\nabla\right)\pi_k$$

$$-\frac{1}{(\pi\delta\sigma)_k}\left[(\pi\dot{\sigma})_{k+1/2}(p_k\hat{\theta}_{k+1/2}-T_k)+(\pi\dot{\sigma})_{k-1/2}(T_k-p_k\hat{\theta}_{k-1/2})\right]$$

$$+C_k \tag{6.6}$$

where $\alpha\equiv c_p\theta(\partial p/\partial\pi)/\sigma$ and $\sigma\equiv(p-p_I)/\pi$. Eqs. (6.4) and (6.6) give

$$C_k=\frac{1}{1+\frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_{pk}}\left\{\left[\left(\frac{\partial q^*}{\partial p}\right)_{Tk}+\frac{\alpha_k}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_{pk}\right]\sigma_k\left(\frac{\partial}{\partial t}+\mathbf{v}_k\cdot\nabla\right)\pi_k\right.$$

$$-\left(\frac{\partial q^*}{\partial T}\right)_{pk}\frac{p_k}{(\pi\delta\sigma)_k}\left[(\pi\dot{\sigma})_{k+1/2}(\hat{\theta}_{k+1/2}-\theta_k)+(\pi\dot{\sigma})_{k-1/2}(\theta_k-\hat{\theta}_{k-1/2})\right]$$

$$\left.+\frac{1}{(\pi\delta\sigma)_k}\left[(\pi\dot{\sigma})_{k+1/2}(\hat{q}_{k+1/2}-q^*_k)+(\pi\dot{\sigma})_{k-1/2}(q^*_k-\hat{q}_{k-1/2})\right]\right\} \tag{6.7}$$

Substituting (6.7) into (6.6), we obtain

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_k\cdot\nabla\right)T_k=\left(\frac{\partial T}{\partial p}\right)_{mk}\left(\frac{\partial}{\partial t}+\mathbf{v}_k\cdot\nabla\right)\pi_k-\frac{1}{1+\frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_{pk}}\frac{1}{(\pi\delta\sigma)_k}\times$$

$$\left\{(\pi\dot{\sigma})_{k+1/2}\left[p_k(\hat{\theta}_{k+1/2}-\theta_k)+\frac{L}{c_p}(\hat{q}_{k+1/2}-q^*_k)\right]\right.$$

$$\left.+(\pi\dot{\sigma})_{k-1/2}\left[p_k(\theta_k-\hat{\theta}_{k-1/2})+\frac{L}{c_p}(q^*_k-\hat{q}_{k-1/2})\right]\right\} \tag{6.8}$$

where

$$\left(\frac{\partial T}{\partial p}\right)_{mk}\equiv\frac{\frac{\alpha_k}{c_p}-\frac{L}{c_p}\left(\frac{\partial q^*}{\partial p}\right)_{Tk}}{1+\frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_{pk}} \tag{6.9}$$

The coefficient of $(\pi\dot{\sigma})_{k+1/2}$ in (6.8) is

$$p_k(\hat{\theta}_{k+1/2}-\theta_k)+\frac{L}{c_p}(\hat{q}_{k+1/2}-q^*_k)$$

$$=\frac{1}{c_p}[(c_p\hat{T}_{k+1/2}+\phi_{k+1/2}+L\hat{q}_{k+1/2})-(c_pT_k+\hat{\phi}_k+Lq^*_k)]$$

$$=\frac{1}{c_p}(\hat{h}_{k+1/2}-h^*_k) \tag{6.10}$$

where (1.13) and (1.14) and the definition of h have been used. Similarly, the coefficient of $(\pi\dot{\sigma})_{k-1/2}$ in (6.8) is

$$\frac{1}{c_p}[(h^*_k-\hat{h}_{k-1/2})] \tag{6.11}$$

Thus (6.8) can be written as

$$(\frac{\partial}{\partial t}+\mathbf{v}_k\cdot\nabla)T_k=(\frac{\partial T}{\partial p})_{mk}(\frac{\partial}{\partial t}+\mathbf{v}_k\cdot\nabla)\pi_k$$

$$-\frac{1}{c_p+L(\frac{\partial q^*}{\partial T})_{pk}}\frac{1}{(\pi\delta\sigma)_k}[(\pi\dot{\sigma})_{k+1/2}(h_{k+1/2}-h_p^*)+(\pi\dot{\sigma})_{k-1/2}(\hat{h}_{k-1/2})] \tag{6.12}$$

The choice of q (or equivalently, h) at the levels remains to be specified.

6.2 Vertical interpolation of moisture and "CICK"

From (6.12) it is clear that a negative $(\pi\dot{\sigma})_{k+1/2}$has a warming effect for $\hat{h}_{k+1/2}>h^*_k$. This may occur even when $h^*_{k+1}<h^*_k$, that is, when no conditional instability exists between the integer levels k+1and k. The same effect can similarly ocuur for a negative $(\pi\dot{\sigma})_{k-1/2}$when $h^*_k>\hat{h}_{k-1/2}$, even when $h^*_k<h^*_{k-1}$. Any moist convective instability produced by such a warming effect is the result merely of a poor choice of $\hat{q}_{k+1/2}$. This type of computational instability is termed as the "conditional instability of computational kind" (CICK). See AM for details.

The CICK can be avoided if the choice of $\hat{q}_{k+1/2}$and thus $\hat{h}_{k+1/2}$ satisfies the followiog requirements when $h^*_{k+1}<h^*_k$:

$$\hat{h}_{k+1/2}<h^*_k \quad \text{when} \quad r_k=1 \tag{6.13}$$

and

$$h^*_{k+1}<\hat{h}_{k+1/2} \quad \text{when} \quad r_{k+1}=1 \tag{6.14}$$

where r_k is the relative humidity of the level k, given by

$$r_k=q_k/q^*_k \tag{6.15}$$

Let us write $\hat{h}_{k+1/2}$ as

$$\begin{aligned}\hat{h}_{k+1/2} &\equiv \hat{s}_{k+1/2} + L\hat{q}_{k+1/2} \\ &= \hat{s}_{k+1/2} + L\hat{r}_{k+1/2}\hat{q}^*_{k+1/2} \\ &= \hat{s}_{k+1/2} + \hat{r}_{k+1/2}(\hat{h}^*_{k+1/2} - \hat{s}_{k+1/2}) \\ &= (1-\hat{r}_{k+1/2})\hat{s}_{k+1/2} + \hat{r}_{k+1/2}\hat{h}^*_{k+1/2} \end{aligned} \tag{6.16}$$

where $\hat{r}_{k+1/2}$ is a properly defined relative humidity at level $k+1/2$, and

$$\hat{s}_{k+1/2} \equiv c_p \hat{T}_{k+1/2} + \phi_{k+1/2} \tag{6.17}$$

Substitutions of (6.16) into (6.13) and (6.14) give

$$(1-\hat{r}_{k+1/2})\hat{s}_{k+1/2} + \hat{r}_{k+1/2}\hat{h}^*_{k+1/2} < h^*_k \quad \text{when} \quad r_k = 1$$

and (6.18)

$$h^*_{k+1} < (1-\hat{r}_{k+1/2})\hat{s}_{k+1/2} + \hat{r}_{k+1/2}\hat{h}^*_{k+1/2} \quad \text{when} \quad r_{k+1} = 1$$

Suppose that $\hat{h}^*_{k+1/2}$ is an interpolation of h^* from the level k and $k+1$ to the level $k+1/2$ that guarantees $h^*_{k+1} < \hat{h}^*_{k+1/2} < h^*_k$ if $h^*_{k+1} < h^*_k$. If we choose $\hat{r}_{k+1/2} = 1$ when either $r_k = 1$ or $r_{k+1} = 1$, the inequalities (6.18) are satisfied, ragardless of the actual form of the interpolation for $h^*_{k+1/2}$ under the condition $h^*_{k+1} < h^*_k$. The form given below satisfies the above requirement.

$$\hat{r}_{k+1/2} = \frac{r_k + r_{k+1} - 2r_k r_{k+1}}{2 - r_k - r_{k+1}} \tag{6.19}$$

(When both r_k and r_{k+1} are 1, $\hat{r}_{k+1/2}$ is set to 1).

The form of the interpolation used to obtain h^* is important, however, in relation to the interpolation chosen for $\hat{s}_{k+1/2}$. Since

$$h^*_k = Lq^*_k + s_k \tag{6.20}$$

an interpolation for $\hat{h}^*_{k+1/2}$ independent of for $\hat{s}_{k+1/2}$ could in theory allow the implicit generation of a negative $\hat{q}^*_{k+1/2}$. To avoid this, the interpolation for $\hat{h}^*_{k+1/2}$ is chosen proportional to that for s_{k+1}:

$$\begin{aligned}\hat{h}^*_{k+1/2} - h^*_k &= A(\hat{s}_{k+1/2} - s_k) \\ h^*_{k+1} - \hat{h}^*_{k+1/2} &= A(s_{k+1} - \hat{s}_{k+1/2})\end{aligned} \tag{6.12}$$

and

$$A \equiv \frac{h^*_{k+1} - h^*_k}{s_{k+1} - s_k} \tag{6.22}$$

Recall that (1.12) and (1.13) give

$$\begin{aligned}\hat{s}_{k+1/2} - s_k &= P_k c_p (\hat{\theta}_{k+1/2} - \theta_k) \\ s_{k+1} - \hat{s}_{k+1/2} &= P_{k+1} c_p (\theta_{k+1} - \hat{\theta}_{k+1/2})\end{aligned} \tag{6.23}$$

Then we obtain

$$s_{k+1} - s_k = P_k c_p (\hat{\theta}_{k+1/2} - \theta_k) + P_{k+1} c_p (\theta_{k+1} - \hat{\theta}_{k+1/2}) \tag{6.24}$$

which determines the denominator of (6.22). Eq. (6.16) gives

$$q_{CICK} \equiv \hat{q}_{k+1/2} = \frac{1}{L} \hat{r}_{k+1/2} (\hat{h}^*_{k+1/2} - \hat{s}_{k+1/2}) \tag{6.25}$$

There is no reason to choose this q, howver, if condensation processes are not involved. For the relatively dry case, it is more important to guarantee that q remains positive or zero. The simple arithmetic averge, $\hat{q}_{k+1/2} = \frac{1}{2}(q_k + q_{k+1})$ is not a good choice. Because, if $q_k = 0$, $q_{k+1} > 0$ and $(\pi\dot{\sigma})_{k+1/2} > 0$, then downward current removes a positive amount from zero. Presumably, the application of (1.20) to water vapor mixing ratio is a better choice for the relatively dry case. It is given by

$$q_{ln} \equiv \hat{q}_{k+1/2} = \frac{\ln q_k - \ln q_{k+1}}{1/q_{k+1} - 1/q_k} \tag{6.26}$$

In fact, with that choice, $\hat{q}_{k+1/2}$ is zero when either q_{k+1} or q_k is zero.

In the present model, the following formula is used as a compromise.

$$\hat{q}_{k+1/2} = \hat{r}_{k+1/2} q_{CICK} + (1 - \hat{r}_{k+1/2}) q_{ln} \tag{6.27}$$

6.3 Vertical differencing of the ozone equation

The ozone equation does not have the difficulty such as CICK mentiond in the previous section. We simply adopt the vertical interpolation (1.20) for ozone mixing ratio.

$$\hat{O}_{3\ k+1/2} = \frac{\ln O_{3\ k} - \ln O_{3\ k+1}}{1/O_{3\ k+1} - 1/O_{3\ k}} \tag{6.28}$$

where O_3 is the mixing ratio of ozone. The vertical differencing scheme for ozone is the same as (6.2) except that $-C_k$ term is replaced by the souce term of ozone due to photochemical processes described in Chapter 12.

6.4 Horizontal differencing

The finite horizontal differencing schemes for both moisture and ozone are the same in the MRI• GCM-I. In the following only the scheme for moisture is shown.

Using eqs. (3.1) through (3.4), the horizontal flux term of the moisture equation (6.1) can be written as

$$\frac{\partial}{\partial t}(\pi_{i,j} q_{i,j}) = -F_{i+1/2,j} \hat{q}_{i+1/2,j} + F_{i-1/2,j} \hat{q}_{i-1/2,j} - G_{i,j+1/2} \hat{q}_{i,j+1/2} + G_{i,j-1/2} \hat{q}_{i,j-1/2} \tag{6.29}$$

+other terms

The $\hat{q}$ is defined by the arithmetic average, for example,

$$\hat{q}_{i+1/2,j}=\frac{1}{2}(q_{i,j}+q_{i+1,j}) \tag{6.30}$$

The difficulty, which comes from the existence of the lower limit of q (that is, $q=0$), also exists in the horizontal differencing. Therefore, $\hat{q}$ is defined as

$$\hat{q}_{i+1/2,j}=\frac{2q_{i,j}q_{i+1,j}}{q_{i,j}+q_{i+1,j}} \tag{6.31}$$

when $F_{i+1/2,j}>0$ and $q_{i,j}<q_{i+1,j}$ or $F_{i+1/2,j}<0$ and $q_{i,j}>q_{i+1,j}$. Note that this form is formally derived from (1.2) if we choose $G(x)=1/x$. Otherwise, eq. (6.30) is used in the present model. $\hat{q}_{i-1/2,j}$, $\hat{q}_{i,j+1/2}$ and $\hat{q}_{i,j-1/2}$ are defined in a similar way.