4. Special treatment near the poles*

4.1 Modifications of the difference equation

As the poles are singular points in the spherical coordinate, velocity components cannot be defined there. Therefore, we let the poles be π-points. The mass at the poles changes through the meridional mass flux, G, at all the v-points surrounding the poles, as shown in Fig.4.1. We let the north pole be j=p, and treat it as if it were a group of points, just for simplifying the computation. Each point has index i and represents the area shaded in Fig.4.1. Defining $\Pi_{i,p}$ and $S_{i,p}$ based on that area, we apply the equation of continuity (4.1) to j=p, omitting all horizontal mass flux terms except $G_{i,p-1/2}$. After computing $\partial\Pi/\partial t$ and S for all i, we take the average.

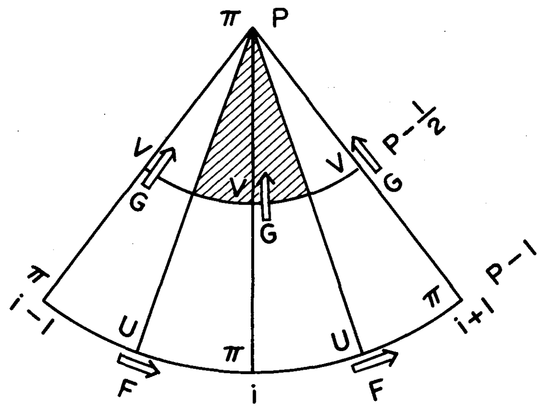

Fig. 4.1 Horizontal indices and location of variables in vicinity of the north pole.

$$\frac{\partial}{\partial t}\sum_i \Pi^{k}_{i,p} - \sum_i G_{i,p-1/2} + \frac{1}{\Delta\sigma_k}\sum_i(\dot{S}^{k+1}_{i,p} - \dot{S}^{k-1}_{i,p}) = 0 \qquad (4.1)$$

The advective terms in the v-momentum equation at the point (i,p−1/2) are given the following form;

$$\frac{\partial}{\partial t}(\Pi^{(v)}v)^{k}_{i,p-1/2} + \frac{1}{2}[F^{*(v)}_{i+1/2,p-1/2}(v_{i,p-1/2} + v_{i,p-1/2})$$

$$-F^{*(v)}_{i-1/2,p-1/2}(v_{i,p-1/2} + v_{i-1,p-1/2})$$

$$-G^{(v)}_{i,p-1}(v_{i,p-1/2} + v_{i,p-3/2})$$

$$-\hat{F}^{(v)}_{i-1/2,p-1}(v_{i,p-1/2} + v_{i-1,p-3/2})$$

$$-\tilde{G}^{(v)}_{i+1/2,p-1}(v_{i,p-1/2} + v_{i+1,p-3/2})]^{k}$$

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* This chapters is prepared by T. Tokioka.

$$+\frac{1}{\Delta\sigma_k}\cdot\frac{1}{2}[\dot{S}^{(v)k+1}(v^{k+2}+v^k)-\dot{S}^{(v)k-1}(v^k+v^{k-2})]_{i,p-1/2} \tag{4.2}$$

The form of $\Pi^{(v)}_{i,p-1/2}$ should be,

$$\Pi^{(v)}_{i,p-1/2}=\frac{1}{8}\{\Pi_{i+1,p}+\Pi_{i+1,p-1}+2(\Pi_{i,p}+\Pi_{i,p-1})+\Pi_{i-1,p}+\Pi_{i-1,p-1}\}$$

$$+\frac{1}{8}(\Pi_{i+1,p}+2\Pi_{i,p}+\Pi_{i-1,p}) \tag{4.3}$$

so that the global sum of $\Pi^{(v)}$ is equal to the global sum of $\Pi^{(u)}$. $\dot{S}^{(v)}_{i,p-1/2}$ is defined, also, by (4.3) by replacing Π with $\dot{S}$. If we replace v in (4.2) by a constant value, it reduces to an analog of the continuity equation. In order that the equation is compatible with the continuity equation (3.1) and (4.1),

$$F^{*(v)}_{i+1/2,p-1/2}=\frac{1}{6}(F^*_{i,p-1}+F^*_{i+1,p-1}) \tag{4.4}$$

In a similar way, advective terms in the u-momentum equation at the grid point (i+1/2,p−1,k), $\Pi^{(u)}_{i+1/2,p-1}$ and $F^{*(u)}_{i,p-1}$ are

$$\frac{\partial}{\partial t}(\Pi^{(u)}u)^k_{i+1/2,p-1}+\frac{1}{2}[F^{*(u)}_{i+1,p-1}(u_{i+3/2,p-1}+u_{i+1/2,p-1})$$

$$-F^{*(u)}_{i,p-1}(u_{i+1/2,p-1}+u_{i-1/2,p-1})$$

$$-G^{(u)}_{i+1/2,p-1/2}(u_{i+1/2,p-1}+u_{i+1/2,p-2})-\tilde{F}^{(u)}_{i,p-3/2}(u_{i+1/2,p-1}+u_{i-1/2,p-2})$$

$$-\tilde{G}^{(u)}_{i+1,p-3/2}(u_{i+3/2,p-2}+u_{i+1/2,p-1})]^k$$

$$+\frac{1}{\Delta\sigma_k}\cdot\frac{1}{2}[\dot{S}^{(u)k+1}(u^{k+2}+u^k)-\dot{S}^{(u)k-1}(u^k+u^{k-2})]_{i+1/2,p-1} \tag{4.5}$$

$$\Pi^{(u)}_{i+1/2,p-1}=\frac{1}{8}\{\Pi_{i,p}+\Pi_{i+1,p}+2(\Pi_{i,p-1}+\Pi_{i+1,p-1})+\Pi_{i,p-2}+\Pi_{i+1,p-2}\}$$

$$+\frac{1}{8}\{3(\Pi_{i,p}+\Pi_{i+1,p})+\Pi_{i,p-1}+\Pi_{i+1,p-1}\} \tag{4.6}$$

$$F^{*(u)}_{i,p-1}=\frac{1}{6}(4F^*_{i,p-1}+F^*_{i,p-2}) \tag{4.7}$$

$\dot{S}^{(u)}_{i+1/2,p-1}$ is readily defined by replacing Π in (4.6) by $\dot{S}$.

4.2 Introduction of averaging operator to selected terms

The grid interval in the east-west direction decreases with the increase of latitude. In order to avoid the use of a small time interval with that decrease, we have introduced an averaging operator in the east-west direction to selected terms, following Arakawa's analysis (AM or AL). The averaging operator is required to the pressure gradient force in the east-west direction and $\partial(\pi\ u/n)/\partial\xi$ in the continuity equation. In order to maintain conservation of kinetic energy in the advective process still after the modification of the continuity equation, several terms in the momentum equations should also be replaced by the smoothed value of them. The superior bars in Chapter 3 are the reminders that those terms should be smoothed somehow in the east-west direction.

The momentum equations can be transformed into the vorticity equation. In order that the vorticity equation thus derived does not have terms with no correspondence in the continuous form of it, the terms with the double superior bars should also be replaced by the smoothed values of them. The averaging operators denoted by both the single and the double superior bars are identical. The different notations are used just to remind that the operator is required from the different reasons.

Let $d\xi$ and $d\eta$ be the longitudinal and latitudinal grid sizes. Arakawa's analysis show that the amplitude of Fourier component of the term in the longitudinal direction with the wavenumber k should be multiplied by the factor S(j,k);

$$S(j,k)=\min(1,\ \frac{d\xi_j}{d\eta}/\sin(\frac{k}{2}d\xi_j)) \tag{4.8}$$

where min (,) is the operation to take the less value between the two in the parenthesis. As the operation is linear, we can define the corresponding operator in the real grid space, which is nothing but the superior bar operator. Actually, approximate forms of the operator is adopted currently for the economy of computation.