3. Horizontal differencing*

3.1 Horizontal grid and indices

Spherical coordinate is adopted in the model. Grids on the sphere are distributed in equal intervals in both longitudinal and latitudinal directions. Winninghoff (1968) has shown that the geostrophic adjustment process depends on how the variables are distributed over the grid points. Among five ways of distributing the dependent variables (see Fiq. 3.1), Scheme C gives the best dispersion relation for inertio-gravity waves, where u and v are velocity components in both "i" and "j" directions, respectively and ϕ is geopotential.

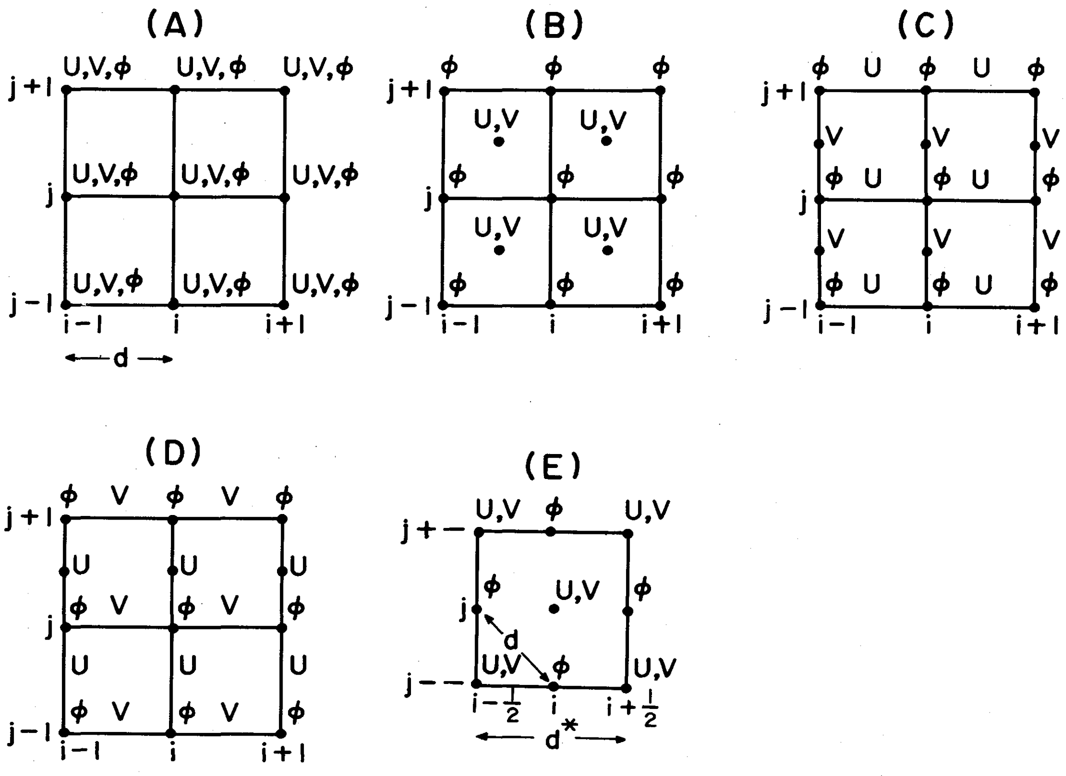

Fig. 3.1 Five ways of distributing variables on the horizontal grids. (Taken from AL) u and v are horizontal components of wind in "i" and "j" directions, respectively. ϕ indicates geopotential.

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* This chapter is prepared by T. Tokioka.

Based on the above results, Scheme C is adopted in distributing variables over the sphere. Indices "i" and "j" are used to indicate grid position in longitudinal and latitudinal directions respectively (see Fig. 3.2). Surface pressure p_s, geopotential ϕ, temperature T, mixing ratios of water vapor q and ozone O_3, and vertical velocity $\dot{\sigma}$ are defined at the π-point in Fig.3.2.

The details of the horizontal differencing described below are the same as that described by AM or AL except for some special treatments near the poles.

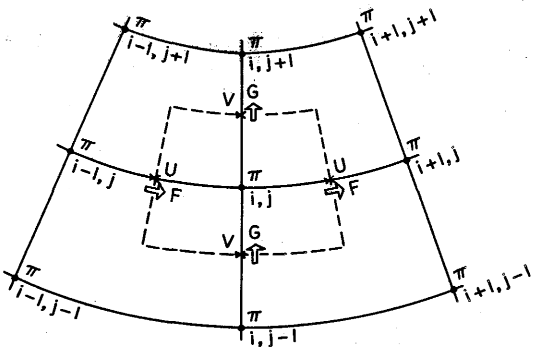

Fig. 3.2 Horizontal indices and location of variables. At π-points, all thermodynamic variables, including water vapor and ozone, are carried. As for the definitions of F and G, see text. (Taken from AL)

3.2 The equation of continuity

The equation of continuity (0.14) is expreseed as follows;

$$\frac{\partial \Pi_{ij}}{\partial t}+F^k_{i+1/2,j}-F^k_{i-1/2,j}+G^k_{i,j+1/2}-G^k_{i,j-1/2}+\frac{1}{\Delta\sigma_k}(\dot{S}^{k+1}_{ij}-\dot{S}^{k-1}_{ij})=0 \tag{3.1}$$

where

$$\Pi\equiv\pi\frac{\Delta\xi\Delta\eta}{mn},\quad F\equiv\pi u\frac{\Delta\eta}{n},\quad G\equiv\pi v\frac{\Delta\xi}{m},\quad \dot{S}\equiv\Pi\dot{\sigma} \tag{3.2}$$

and

$$\xi=\lambda,\quad \eta=\varphi,\quad \frac{1}{m}=a\cos\varphi \text{ and } \frac{1}{n}=a \tag{3.3}$$

The mass flux F and G are defined as follows;

$$\left.\begin{aligned} F^k_{i+1/2,j}&=\frac{1}{2}\overline{\left(u\frac{\Delta\eta}{n}\right)}^k_{i+1/2,j}(\pi_{i+1,j}+\pi_{i,j}) \\ G^k_{i,j+1/2}&=\frac{1}{2}\overline{\overline{\left(v\frac{\Delta\xi}{m}\right)}}^k_{i,j+1/2}(\pi_{i,j+1}+\pi_{i,j}) \end{aligned}\right\} \tag{3.4}$$

For the time being, ignore the superior bar operators in (3.4), which are linear smoothing operators in ξ.

3.3 The pressure gradient force

The pressure gradient force in the ξ -direction is

$$-\frac{\pi}{n}\left[\frac{\partial\phi}{\partial\xi}+\sigma\alpha\frac{\partial\pi}{\partial\xi}\right]$$

For the first term, we choose the form

$$-\left(\frac{\pi}{n}\frac{\partial\phi}{\partial\xi}\right)^{k}_{i+1/2,j}=-\frac{1}{\Delta\xi\Delta\eta}\left(\frac{\Delta\eta}{n}\right)_{j}\overline{\frac{1}{2}(\pi_{i+1,j}+\pi_{i,j})(\hat{\phi}^{k}_{i+1,}-\hat{\phi}^{k}_{i,j})} \tag{3.5}$$

As for the second term, we choose the form

$$-\left(\frac{\pi\sigma\alpha}{n}\frac{\partial\pi}{\partial\xi}\right)^{k}_{i+1/2,j}=-\frac{1}{\Delta\xi\Delta\eta}\left(\frac{\Delta\eta}{n}\right)_{j}\cdot\overline{\frac{1}{2}\{(\pi\sigma\alpha)^{k}_{i+1,j}+(\pi\sigma\alpha)^{k}_{i,j}\}(\pi_{i+1,j}-\pi_{i,j})} \tag{3.6}$$

Rearranging the right hand side of (3.5), we confirm that both (3.5) and (3.6), with the help of (1.7), guarantee the relation (1.6) at each grid point.

Similarly, the pressure gradient force in the η-direction is

$$-\left(\frac{\Delta\xi}{m}\right)_{j+1/2}\cdot\frac{1}{2}\left[\overline{\overline{(\pi_{i,j+1}+\pi_{i,j})(\hat{\phi}^{k}_{i,j+1}-\hat{\phi}^{k}_{i,j})}}+\overline{\overline{\{(\pi\sigma\alpha)^{k}_{i,j+1}+(\pi\sigma\alpha)^{k}_{i,j}\}(\pi_{i,j+1}-\pi_{i,j})}}\right] \tag{3.7}$$

3.4 Kinetic energy generation and the first law of thermodynamics

The contribution of the pressure gradient force to the kinetic energy generation

$$\frac{\partial}{\partial t}\left(\Pi\cdot\frac{1}{2}u^{2}\right)^{k}_{i+1/2,j}$$

is obtained by multiplying (3.5) and (3.6) by $u^{k}_{i+1/2,j}$. Then the kinetic energy generation is

$$-\frac{1}{2}\left(u\frac{\Delta\eta}{n}\right)^{k}_{i+1/2,j}\left[\overline{(\pi_{i+1,j}+\pi_{i,j})(\hat{\phi}^{k}_{i+1,j}-\hat{\phi}^{k}_{i,j})}+\overline{\{(\pi\sigma\alpha)^{k}_{i+1,j}+(\pi\sigma\alpha)^{k}_{i,j}\}(\pi_{i+1,j}-\pi_{i,j})}\right] \tag{3.8}$$

As the superior bar indicates a linear smoothing operator in ξ, the summation of (3.8) over i is identical to that of the following,

$$-\frac{1}{2}\overline{\left(u\frac{\Delta\eta}{n}\right)^{k}_{i+1/2,j}}\left[(\pi_{i+1,j}+\pi_{ij})(\hat{\phi}^{k}_{i+1,j}-\hat{\phi}^{k}_{i,j})+\{(\pi\sigma\alpha)^{k}_{i+1,j}+(\pi\sigma\alpha)^{k}_{i,j}\}(\pi_{i+1,j}-\pi_{ij})\right] \tag{3.9}$$

This can be written as;

$$(3.9)=-F^{k}_{i+\frac{1}{2},j}(\hat{\phi}^{k}_{i+1,j}-\hat{\phi}^{k}_{i,j})-\frac{1}{2}\overline{(u\frac{\Delta\eta}{n})}^{k}_{i+\frac{1}{2},j}\{(\pi\sigma\alpha)^{k}_{i+1,j}+(\pi\sigma\alpha)^{k}_{i,j}\}(\pi_{i+1,j}-\pi_{i,j}) \quad (3.9)'$$

Therefore we can show that

$$\sum_i(3.9)=\sum_i(F^{k}_{i+\frac{1}{2},j}-F^{k}_{i-\frac{1}{2},j})\hat{\phi}^{k}_{i,j}-\frac{1}{4}\overline{(u\frac{\Delta\eta}{n})}^{k}_{i+\frac{1}{2},j}\{(\pi\sigma\alpha)^{k}_{i+1,j}+(\pi\sigma\alpha)^{k}_{i,j}\}(\pi_{i+1,j}-\pi_{ij})$$

$$-\frac{1}{4}\overline{(u\frac{\Delta\eta}{n})}^{k}_{i-\frac{1}{2},j}\{(\pi\sigma\alpha)^{k}_{i,j}+(\pi\sigma\alpha)^{k}_{i-1,j}\}(\pi_{i,j}-\pi_{i-1,j}) \quad (3.10)$$

Similarly, the contribution of the pressure gradient force to

$$\frac{\partial}{\partial t}(\Pi\frac{1}{2}v^2)$$

is given by

$$\sum_{i,j}(G^{k}_{i,j+\frac{1}{2}}-G^{k}_{i,j-\frac{1}{2}})\hat{\phi}^{k}_{i,j}-\frac{1}{4}\overline{\overline{(v\frac{\Delta\xi}{m})}}^{k}_{i,j+\frac{1}{2}}\{(\pi\sigma\alpha)^{k}_{i,j+1}+(\pi\sigma\alpha)^{k}_{i,j}\}(\pi_{i,j+1}-\pi_{i,j})$$

$$-\frac{1}{4}\overline{\overline{(v\frac{\Delta\xi}{m})}}^{k}_{i,j-\frac{1}{2}}\{(\pi\sigma\alpha)^{k}_{i,j}+(\pi\sigma\alpha)^{k}_{i,j-1}\}(\pi_{i,j}-\pi_{i,j-1}) \quad (3.11)$$

The sum of both (3.10) and (3.11), with the use of continuity equation (3.1), is transformed into the following form;

$$\text{kinetic energy generation}=-\sum_{i,j}\frac{1}{\Delta\sigma_k}\left[(\dot{S}^{k+1}_{i,j}+\sigma_{k+1}\frac{\partial\Pi^k}{\partial t})\phi^{k+1}_{i,j}-(\dot{S}^{k-1}_{i,j}+\sigma_{k-1}\frac{\partial\Pi^k}{\partial t})\phi^{k-1}_{i,j}\right]$$

$$-\sum\Pi^{k}_{i,j}(\omega\alpha)^{k}_{i,j}$$

provided that $(\omega\alpha)^{k}_{i,j}$ is defined by

$$\Pi^{k}_{i,j}(\omega\alpha)^{k}_{i,j}=(\pi\sigma\alpha)^{k}_{i,j}\frac{\partial\Pi^k}{\partial t}+\frac{1}{4}\{(\pi\sigma\alpha)_{i+1,j}+(\pi\sigma\alpha)_{i,j}\}\overline{(u\frac{\Delta\eta}{n})}_{i+\frac{1}{2},j}(\pi_{i+1,j}-\pi_{i,j})$$

$$+\frac{1}{4}\{(\pi\sigma\alpha)_{i,j}+(\pi\sigma\alpha)_{i-1,j}\}\overline{(u\frac{\Delta\eta}{n})}_{i-\frac{1}{2},j}(\pi_{i,j}-\pi_{i-1,j})$$

$$+\frac{1}{4}\{(\pi\sigma\alpha)_{i,j+1}+(\pi\sigma\alpha)_{i,j}\}\overline{\overline{(v\frac{\Delta\xi}{m})}}_{i,j+\frac{1}{2}}(\pi_{i,j+1}-\pi_{i,j})$$

$$+\frac{1}{4}\{(\pi\sigma\alpha)_{i,j}+(\pi\sigma\alpha)_{i,j-1}\}\overline{\overline{(v\frac{\Delta\xi}{m})}}_{i,j-\frac{1}{2}}(\pi_{i,j}-\pi_{i,j-1})$$

$$-\frac{1}{\Delta\sigma_k}\{\dot{S}^{k+1}_{i,j}(\phi^{k+1}_{i,j}-\hat{\phi}^{k}_{i,j})+\dot{S}^{k-1}_{i,j}(\hat{\phi}^{k}_{i,j}-\phi^{k-1}_{i,j})\} \quad (3.13)$$

This expression may be compared with the definition given by (1.9).

Thus the thermodynamic energy equation (1.11) may be written as

$$\frac{\partial}{\partial t}(\Pi_{i,j}T^{k}_{i,j})+F^{k}_{i+½,j}\frac{T^{k}_{i+1,j}+T^{k}_{i,j}}{2}-F^{k}_{i-½,j}\frac{T^{k}_{i,j}+T^{k}_{i-1,j}}{2}+G^{k}_{i,j+½}\frac{T^{k}_{i,j+1}+T^{k}_{i,j}}{2}-G^{k}_{i,j-½}\frac{T^{k}_{i,j}+T^{k}_{i,j-1}}{2}$$

$$+\frac{1}{\Delta\sigma_{k}}[\dot{S}^{k+1}_{i,j}P^{k}_{i,j}\hat{\theta}^{k+1}_{i,j}-\dot{S}^{k-1}_{i,j}P^{k}_{i,j}\hat{\theta}^{k-1}_{i,j}]=\frac{1}{c_{p}}\Big[(\pi\sigma\alpha)^{k}_{i,j}\frac{\partial\Pi_{i,j}}{\partial t}$$

$$+\frac{1}{4}\overline{\left(u\frac{\Delta\eta}{n}\right)}_{i+½,j}\{(\pi\sigma\alpha)^{k}_{i+1,j}+(\pi\sigma\alpha)^{k}_{i,j}\}(\pi_{i+1,j}-\pi_{i,j})$$

$$+\frac{1}{4}\overline{\left(u\frac{\Delta\eta}{n}\right)}_{i-½,j}\{(\pi\sigma\alpha)^{k}_{i,j}+(\pi\sigma\alpha)^{k}_{i-1,j}\}(\pi_{i,j}-\pi_{i-1,j})$$

$$+\frac{1}{4}\overline{\overline{\left(v\frac{\Delta\xi}{m}\right)}}_{i,j+½}\{(\pi\sigma\alpha)^{k}_{i,j+1}+(\pi\sigma\alpha)^{k}_{i,j}\}(\pi_{i,j+1}-\pi_{i,j})$$

$$+\frac{1}{4}\overline{\overline{\left(v\frac{\Delta\xi}{m}\right)}}_{i,j-½}\{(\pi\sigma\alpha)^{k}_{i,j}+(\pi\sigma\alpha)^{k}_{i,j-1}\}(\pi_{i,j}-\pi_{i,j-1})$$

$$+\Pi_{i,j}Q^{k}_{i,j}\Big] \tag{3.14}$$

3.5 Momentum fluxes

The expression of momentum fluxes in the finite difference form strictly follows the one developed by Arakawa (see AM or AL). We choose for

$$\frac{\partial}{\partial t}\left(\pi\frac{\Delta\xi\Delta\eta}{mn}u\right)+\Delta\xi\frac{\partial}{\partial\xi}\left(\pi u\frac{\Delta\eta}{n}u\right)+\Delta\eta\frac{\partial}{\partial\eta}\left(\pi v\frac{\Delta\xi}{m}u\right)+\frac{\partial}{\partial\sigma}\left(\pi\dot{\sigma}\frac{\Delta\xi\Delta\eta}{mn}u\right)$$

the form;

$$\frac{\partial}{\partial t}(\Pi^{(u)}_{i,j}u^{k}_{i,j})+\frac{1}{2}[F^{(u)}_{i+½,j}(u_{i+1,j}+u_{i,j})-F^{(u)}_{i-½,j}(u_{i,j}+u_{i-1,j})$$

$$+G^{(u)}_{i,j+½}(u_{i,j+1}+u_{i,j})-G^{(u)}_{i,j-½}(u_{i,j}+u_{i,j-1})$$

$$+\widetilde{F}^{(u)}_{i+½,j+½}(u_{i+1,j+1}+u_{i,j})-\widetilde{F}^{(u)}_{i-½,j-½}(u_{i,j}+u_{i-1,j-1})$$

$$+\widetilde{G}^{(u)}_{i-½,j+½}(u_{i-1,j+1}+u_{i,j})-\widetilde{G}^{(u)}_{i+½,j-½}(u_{ij}+u_{i+1,j-1})]$$

$$+\frac{1}{\Delta\sigma_{k}}\cdot\frac{1}{2}[\dot{S}^{(u)k+1}_{i,j}(u^{k+2}_{i,j}+u^{k}_{i,j})-\dot{S}^{(u)k-1}_{i,j}(u^{k}_{i,j}+u^{k-2}_{i,j})] \tag{3.15}$$

where $\Pi^{(u)}$, $S^{(u)}$, $F^{(u)}$, $G^{(u)}$, $\widetilde{F}^{(u)}$ and $\widetilde{G}^{(u)}$ are not defined yet. When u is constant both in space and time, (3.15) should be zero. Then we get a continuity equation;

$$\frac{\partial \Pi^{(u)}_{ij}}{\partial t} + (F^{(u)}_{i+\frac{1}{2},j} - F^{(u)}_{i-\frac{1}{2},j} + G^{(u)}_{i,j+\frac{1}{2}} - G^{(u)}_{i,j-\frac{1}{2}})^k + (\widetilde{F}^{(u)}_{i+\frac{1}{2},j+\frac{1}{2}} - \widetilde{F}^{(u)}_{i-\frac{1}{2},j-\frac{1}{2}} + \widetilde{G}^{(u)}_{i-\frac{1}{2},j+\frac{1}{2}} - \widetilde{G}^{(u)}_{i+\frac{1}{2},j-\frac{1}{2}})^k + \frac{1}{\Delta\sigma_k}(\dot{S}^{(u)k+1}_{i,j} - \dot{S}^{(u)k-1}_{i,j}) = 0 \quad (3.16)$$

Following AM or AL, we let

$$\left.\begin{aligned} F^{(u)}_{i+\frac{1}{2},j} &= \frac{1}{6}(F^*_{i+\frac{1}{2},j+1} + 2F^*_{i+\frac{1}{2},j} + F^*_{i+\frac{1}{2},j-1}) \\ G^{(u)}_{i,j+\frac{1}{2}} &= \frac{1}{6}(G^*_{i+\frac{1}{2},j} + G^*_{i+\frac{1}{2},j+1} + G^*_{i-\frac{1}{2},j} + G^*_{i-\frac{1}{2},j+1}) \\ \widetilde{F}^{(u)}_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{12}(G^*_{i+\frac{1}{2},j} + G^*_{i+\frac{1}{2},j+1} + F^*_{i+\frac{1}{2},j} + F^*_{i+\frac{1}{2},j+1}) \\ \widetilde{G}^{(u)}_{i-\frac{1}{2},j-\frac{1}{2}} &= \frac{1}{12}(G^*_{i-\frac{1}{2},j} + G^*_{i-\frac{1}{2},j+1} - F^*_{i-\frac{1}{2},j} - F^*_{i-\frac{1}{2},j+1}) \end{aligned}\right\} \quad (3.17)$$

where F^* and G^* are defined by

$$F^*_{i,j} = \frac{1}{2}(F_{i+1/2,j} + F_{i-1/2,j})$$

$$G^*_{i,j} = \frac{1}{2}(G_{i,j+1/2} + G_{i,j-1/2}) \quad (3.18)$$

With the use of (3.17) and (3.18), it is shown that (3.16) is identical to (3.1) provided that

$$\Pi^{(u)}_{i+\frac{1}{2},j} = \frac{1}{8}[\Pi_{i+1,j+1} + \Pi_{i,j+1} + \Pi_{i,j-1} + \Pi_{i+1,j-1} + 2(\Pi_{i+1,j} + \Pi_{i,j})] \quad (3.19)$$

and

$$\dot{S}^{(u)}_{i+\frac{1}{2},j} = \frac{1}{8}[\dot{S}_{i+1,j+1} + \dot{S}_{i,j+1} + \dot{S}_{i,j-1} + \dot{S}_{i+1,j-1} + 2(\dot{S}_{i+1,j} + \dot{S}_{i,j})] \quad (3.20)$$

For v-component, we use a form identical to (3.15), with u replaced by v. Corresponding to (3.17), (3.19) and (3.20), we let

$$F^{(v)}_{i+\frac{1}{2},j} = \frac{1}{6}(F^*_{i+1,j+\frac{1}{2}} + F^*_{i+1,j-\frac{1}{2}} + F^*_{i,j+\frac{1}{2}} + F^*_{i,j-\frac{1}{2}})$$

$$G^{(v)}_{i,j+\frac{1}{2}} = \frac{1}{6}(G^*_{i+1,j+\frac{1}{2}} + 2G^*_{i,j+\frac{1}{2}} + G^*_{i,j-\frac{1}{2}})$$

$$\widetilde{F}^{(v)}_{i+\frac{1}{2},j+\frac{1}{2}}=\frac{1}{12}(G^*_{i+1,j+\frac{1}{2}}+G^*_{i,j+\frac{1}{2}}+F^*_{i+1,j+\frac{1}{2}}+F^*_{i,j+\frac{1}{2}}) \tag{3.21}$$

$$\widetilde{G}^{(v)}_{i-\frac{1}{2},j+\frac{1}{2}}=\frac{1}{12}(G^*_{i,j+\frac{1}{2}}+G^*_{i-1,j+\frac{1}{2}}-F^*_{i,j+\frac{1}{2}}-F^*_{i-1,j+\frac{1}{2}})$$

$$\Pi^{(v)}_{i,j+\frac{1}{2}}=\frac{1}{8}[\Pi_{i+1,j+1}+\Pi_{i+1,j}+\Pi_{i-1,j+1}+\Pi_{i-1,j}+2(\Pi_{i,j+1}+\Pi_{i,j})] \tag{3.22}$$

$$\dot{S}^{(v)}_{i,j+\frac{1}{2}}=\frac{1}{8}[\dot{S}_{i+1,j+1}+\dot{S}_{i+1,j}+\dot{S}_{i-1,j+1}+\dot{S}_{i-1,j}+2(\dot{S}_{i,j+1}+\dot{S}_{i,j})] \tag{3.23}$$

In the limit of two-dimensional non-divergent flow, the flux form described above guarantees the conservation of enstrophy as well as kinetic energy as shown by A, AM or AL.

3.6 Coriolis force

Coriolis force plus the metric term which contributes to $\frac{\partial}{\partial t}(\Pi u)$ is

$$\left[f\frac{\Delta\xi\Delta\eta}{mn}-u\Delta\xi\Delta\eta\frac{\partial}{\partial\eta}\left(\frac{1}{m}\right)\right]\pi v \tag{3.24}$$

and the Coriolis force which contributes to $\frac{\partial}{\partial t}(\Pi v)$ is

$$-\left[f\frac{\Delta\xi\Delta\eta}{mn}-u\Delta\xi\Delta\eta\frac{\partial}{\partial\eta}\left(\frac{1}{m}\right)\right]\pi u \tag{3.25}$$

Defining $C^k_{i,j}$ at π-point as follows,

$$C^k_{i,j}=f_j\left(\frac{\Delta\xi\Delta\eta}{mn}\right)_j-\frac{1}{2}(u_{i+\frac{1}{2},j}+u_{i-\frac{1}{2},j})^k\left\{\left(\frac{\Delta\xi}{m}\right)_{j+1/2}-\left(\frac{\Delta\xi}{m}\right)_{j-1/2}\right\} \tag{3.26}$$

we express (3.24), at u-point $(i+\frac{1}{2},j)$, in the following way,

$$\frac{1}{4}\left[\pi_{i+1,j}C^k_{i+1,j}(v_{i+1,j+1/2}+v_{i+1,j-1/2})^k+\pi_{i,j}C^k_{i,j}(v_{i,j+1/2}+v_{i,j-1/2})^k\right] \tag{3.27}$$

(3.25), at v-point $(i,j+\frac{1}{2})$, is expressed as

$$-\frac{1}{4}\left[\pi_{i,j+1}C^k_{i,j+1}(u_{i+1/2,j+1}+u_{i-1/2,j+1})^k+\pi_{i,j}C^k_{i,j}(u_{i+1/2,j}+u_{i-1/2,j})^k\right] \tag{3.28}$$

Note that exact cancellation of kinetic energy generation through Coriolis force is guaranteed by the forms (3.27) and (3.28).