1. Vertical differencing*

1.1 The vertical coordinate and the vertical index

The model atmosphere is discretized by constant σ-levels. There are eight choices of distributing variables on the vertical levels as shown in Fig.1.1. Tokioka(1978) studied this problem from the standpoint of describing vertical dispersions of wawes. Here we follow his results and adopt Scheme C'. Horizontal wind **v** and temperature T are defined at odd levels, and geopotential ϕ and $\dot{\sigma}$ (individual time derivative of σ) are defind at even levels. $\hat{\mathbf{v}}$, $\hat{T}$ and $\hat{\phi}$ in Fig.1.1 indicate values interpolated in some ways from the non-hat values of them.

The vertical indices are given consecutively from the top to the surface as shown in Fig. 1.2. We locate vertical levels above $p=p_I$ in equal interval in ln p, again following Tokioka's analysis(1978), for best simulation of internal waves. Currently, we have two versions of the model. One is the 5-level tropospheric model, where $p_t=p_I=100$mb and $\sigma_2=0.111111$, $\sigma_4=0.333333$, $\sigma_6=0.555556$, $\sigma_8=0.777778$, $\sigma_{10}=1.0$ (see Fig.1.2(a)). The other is the 12-level model,

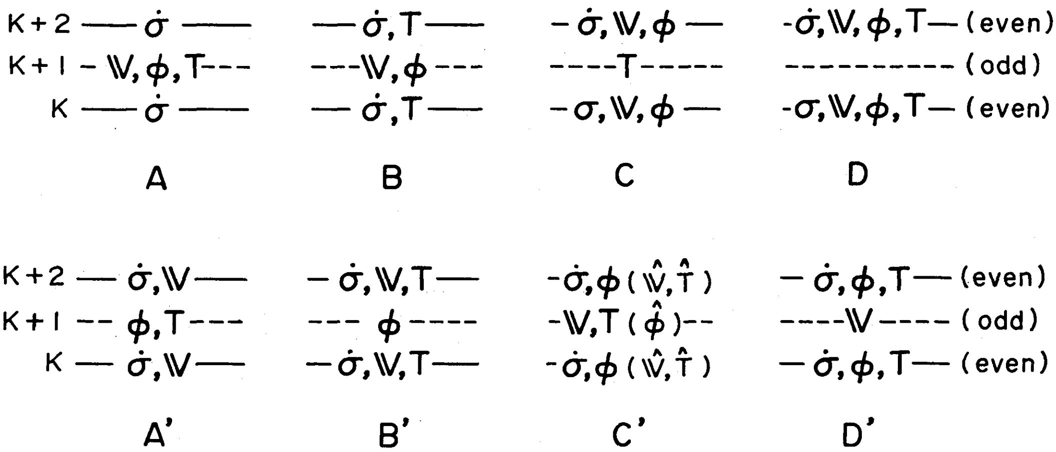
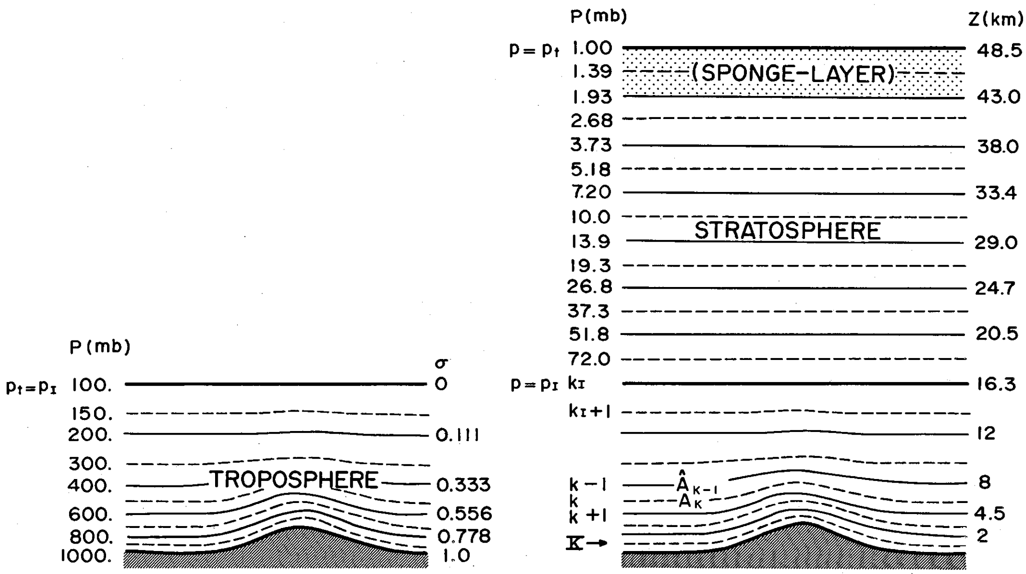


Fig. 1.1 Eight choices of distributing variables on vertical levels. Levels with solid and dashed lines are called as "even" and "odd" levels, respectively. v is horizontal wind vector ; $\dot{\sigma}$, vertical σ-velocity ; T, temperature ; ϕ, geopotential. ^ is a reminder that the variable is an interpolated value from the non-hat ones. Scheme C' is adopted in the MRI・GCM-I.

*This chapter is prepared by T. Tokioka.



(a) (b)

Fig. 1.2 The vertical indices and the position of vertical levels. (a) Five layer version (5L-MRI・GCM-I) and (b) twelve layer version (12L-MRI・GCM-I). As in Fig. 1.1, solid and dashed lines indicate even and odd levels, respectively. The lowest odd level is indicated as K. Approximate positions of the levels in km is shown when the surface pressure is 1000mb.

where $p_t = 1$mb and $p_I = 100$mb. The model atmosphere between $p = p_t$ and p_I is discretized into seven levels in equal interval in ln p (see Fig.1.2(b)). The model structure below the level $p = p_I$ is the same as that of the 5-level model.

1.2 Flux form of a variable

Let A_k be a variable A defined at the odd level k and $\hat{A}_{k+1}$, an interpolated value of A at the even level k+1. We introduce a notation

$$\frac{D}{Dt}(\pi_k * A_k) \equiv \frac{\partial}{\partial t}(\pi_k A_k) + \nabla\cdot(\pi_k \mathbf{v}_k A_k) + \frac{1}{\Delta\sigma_k}\left[(\pi\dot{\sigma})_{k+1}\hat{A}_{k+1} - (\pi\dot{\sigma})_{k-1}\hat{A}_{k-1}\right] \qquad (1.1)$$

where ∇ is the horizontal divergence operator. This is the flux form of variable A and conserves mass weighted integral of A under the vertical boundary condition $\dot{\sigma}_0 = \dot{\sigma}_{K+1} = 0$. We define $\hat{A}_{k+1}$ as

$$\hat{A}_{k+1}=\frac{(G'_{k+2}A_{k+2}-G_{k+2})-(G'_k A_k-G_k)}{G'_{k+2}-G'_k} \tag{1.2}$$

where $G_k \equiv G(A_k)$ is an arbitrary function of variable A_k and $G'_k \equiv dG(A_k)/dA_k$. We can further conserve mass weighted integral of G by use of this form (Arakawa, 1972; Arakawa and Mintz, 1974; or Arakawa and Lamb, 1977)**.

1.3 The equation of continuity

The equation of continuity, (0.14), may be expressed in a discretized form as

$$\frac{D}{Dt}(\pi_k * 1)=0. \tag{1.3}$$

1.4 The acceleration term

The acceleration term in the momentum equation may be expressed as

$$\frac{D}{Dt}(\pi_k * \mathbf{v}_k) \tag{1.4}$$

In order to conserve kinetic energy in the process of vertical advections, we define

$$\hat{\mathbf{v}}_{k+1}=\frac{1}{2}(\mathbf{v}_k+\mathbf{v}_{k+2}), \tag{1.5}$$

This form is obtained from Eq.(1.2) by setting $G(A)=A^2$.

1.5 The pressure gradient force

We introduce the pressure gradient force at the odd level k as

$$\nabla(\pi_k\hat{\phi}_k)-\frac{1}{\Delta\sigma_k}(\phi_{k+1}\sigma_{k+1}-\phi_{k-1}\sigma_{k-1})\nabla\pi_k \tag{1.6}$$

so that no spurious acceleration of a circulation may occur even in a discretized model with topography (see A, AM or AL). Keeping in mind the identity $\nabla(\pi\phi)-\partial(\phi\sigma)/\partial\sigma\cdot\nabla\pi\equiv\pi\nabla\phi+\pi\sigma\alpha\nabla\pi$, we define $(\sigma\alpha)_k$ by

$$\pi_k(\sigma\alpha)_k=\hat{\phi}_k-\frac{1}{\Delta\sigma_k}(\phi_{k+1}\sigma_{k+1}-\phi_{k-1}\sigma_{k-1}) \tag{1.7}$$

where α is the specific volume.

Multplying $-\mathbf{v}_k$ to Eq.(1.6) and rearranging the terms, we arrive at

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** These papers are abbreviated hereafter as A, AM and AL, respectively.

$$-\mathbf{v}_k\cdot[\text{Eq.(1.6)}]=-\nabla\cdot(\pi_k\mathbf{v}_k\hat{\phi}_k)-\frac{1}{\Delta\sigma_k}[\{(\pi\dot{\sigma})_{k+1}+\sigma_{k+1}\frac{\partial\pi_k}{\partial t}\}\phi_{k+1}$$

$$-\{(\pi\dot{\sigma})_{k-1}+\sigma_{k-1}\frac{\partial\pi_k}{\partial t}\}\phi_{k-1}]-\pi_k(\omega\alpha)_k \tag{1.8}$$

where

$$(\omega\alpha)_k=(\sigma\alpha)_k(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla)\pi_k-\frac{1}{\pi_k\Delta\sigma_k}\{(\pi\dot{\sigma})_{k+1}(\phi_{k+1}-\hat{\phi}_k)$$
$$-(\pi\dot{\sigma})_{k-1}(\hat{\phi}_k-\phi_{k-1})\} \tag{1.9}$$

This is just a definition of $(\omega\alpha)_k$ based on the identity

$$-\mathbf{v}\cdot[\nabla(\phi\pi)-\frac{\partial(\phi\sigma)}{\partial\sigma}\nabla\pi]=-\nabla\cdot(\pi\mathbf{v}\phi)-\frac{\partial}{\partial\sigma}(\pi\dot{\sigma}\phi)-\frac{\partial}{\partial\sigma}(\phi\sigma)\frac{\partial\pi}{\partial t}-\pi\alpha\omega$$

1.6 The first law of thermodynamics

If we define temperature at the level k by

$$\left.\begin{aligned}T_k&=\theta_k\cdot P_k\\ P_k&=P(p_{k+1},\ p_{k-1})\end{aligned}\right\} \tag{1.10}$$

the following enthalpy epuation is derived as a finite difference analog of (0.26);

$$\frac{D}{Dt}(\pi_k * c_pT_k)=\pi_kc_pT_k\frac{\partial\ln P_k}{\partial\pi_k}(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla)\pi_k$$

$$+\frac{1}{\Delta\sigma_k}[(\pi\dot{\sigma})_{k+1}c_p(\hat{T}_{k+1}-P_k\hat{\theta}_{k+1})-(\pi\dot{\sigma})_{k-1}c_p(P_k\hat{\theta}_{k-1}-\hat{T}_{k-1})]+\pi_kQ_k \tag{1.11}$$

where P_k is an analog to $(p/p_0)^\kappa$ for the level k, $\kappa=R/c_p$, and p_0 is a reference pressure. (1. 11) is identical to the expression

$$\frac{D}{Dt}(\pi_k * \theta_k)=\frac{Q}{c_pP_k}$$

1.7 Total energy conservation and the hydrostatic equation

To conserve total energy in an adiabatic and non-dissipative process, the r.h.s. of (1.11) except the last term should be identical to $\pi_k(\omega\alpha)_k$ defined by (1.9). Thus we require

$$c_pT_k\partial\ln P_k/\partial\pi_k=(\sigma\alpha)_k \tag{1.12}$$

$$\left.\begin{aligned}c_p(\hat{T}_{k+1}-P_k\hat{\theta}_{k+1})&=\hat{\phi}_k-\phi_{k+1}\\ c_p(P_k\hat{\theta}_{k-1}-\hat{T}_{k-1})&=\phi_{k-1}-\hat{\phi}_k\end{aligned}\right\} \tag{1.30}$$

Eq.(1.12) is required only for $k>k_I$, because π_k is constant above the level $k=k_I$. Both (1.12) and (1.13) are analogs of hydrostatic relation in discretized form.

The scheme described so far conserves momentum, kinetic energy, potential enthalpy and $G(\theta)$ (provided that $\hat{\theta}_{k+1}$ is determined by (1.2)) through the vertical advective process as well as mass itself. The total energy is also conserved by use of the hydrostatic relations (1.12) and (1.13), and no spurious acceleration of a circulation occurs through the pressure gradient force. In the above formulations, there remain several freedoms. They are;

i) the functional form of $G(\theta)$

ii) the functional form of $P(p_{k+1}, p_{k-1})$

iii) a hydrostatic relation that determines either ϕ_{kI} or $\hat{T}_{kI}$.

iv) a hydrostatic relation that determines either ϕ_{k+1} or $\hat{T}_{k+1}$ for $k<k_I$-3.

These freedoms are eliminated, after Tokioka's (1978) study, as follows;

i) $G(\theta)=\ln\theta$ (1.14)

ii) $P_k=(p_k^*/p_0)^\kappa,\ (p_k^*)^a=\dfrac{1}{1+a}\cdot\dfrac{p_{k+1}^{a+1}-p_{k-1}^{a+1}}{p_{k+1}-p_{k-1}},\ a=0.2$ (1.15)

iii) $\phi_{kI}-\hat{\phi}_{kI+1}=c_pT_{kI+1}\cdot\partial\ln P_{kI+1}/\partial p_{kI}\cdot(p_{kI+2}-p_{kI})$ (1.16)

iv) $\phi_{k+1}-\phi_{k-1}=-RT_k(\ln p_{k+1}-\ln p_{k-1}),\ k<k_I-3$ (1.17)

The functional form (1.14) is required to describe vertical propagation of waves properly. The use of (1.14) gives us additional advantage (see AM or AL), i.e., the conservation of entropy because of (1.2) and exact thickness between the even levels for the wide range of stratification including isentropic and polytropic cases.

It may be useful, for the later convenience, to introduce the following expressions of hydrostatic relation, which are equivalent to (1.12) and (1.13),

$$\hat{\phi}_{k+2}-\hat{\phi}_k=-c_p\hat{\theta}_{k+1}(P_{k+2}-P_k) \quad (1.18)$$

$$\hat{\phi}_K=\phi_s+\sum_{k=kI+1}^{K}{}' \pi_k c_p T_k\frac{\partial\ln P_k}{\partial\pi_k}-\sum_{k=kI+1}^{K-2}{}' \sigma_{k+1}c_p\hat{\theta}_{k+1}(P_{k+2}-P_k) \quad (1.19)$$

where

$$\hat{\theta}_{k+1}=\frac{\ln(\theta_k/\theta_{k+2})}{1/\theta_{k+2}-1/\theta_k} \quad (1.20)$$

Σ' in (1.19) represents a summation over odd k.