0. Governing equations in the continuous form*

0.1 The vertical coordinate

The vertical coordinate adopted in the model is a combination of the σ-coordinate and the pressure coordinate. The σ-coordinate is used below the level $p=p_I$ (see Fig.0.1), while p-coordinate above that level. The vertical coordinate is defined as

$$\sigma=\frac{p-p_I}{\pi}, \qquad (0.1)$$

$$\pi=\begin{cases} p_s-p_I & \text{for} \quad p\geq p_I \\ p_I-p_t & \text{for} \quad p<p_I \end{cases} \qquad (0.2)$$

where p_s and p_t are the pressures at the lower (surface) and the upper (top) boundaries of the model, respectively. Therefore $\sigma=-1, 0, 1$ at $p=p_t, p_I, p_s$. As π is constant above the level $p=p_I$, the σ coordinate defined by Eq.(0.1) is nothing but the normalized p-coordinate.

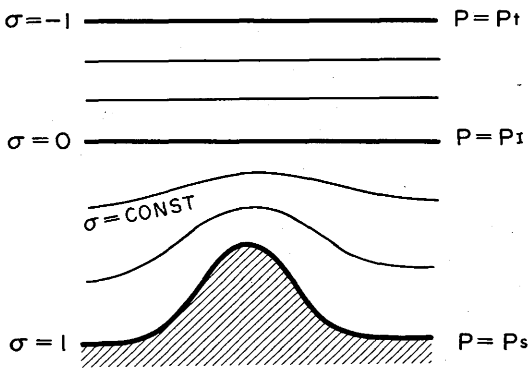

Fig. 0.1 The vertical coordinate σ adopted in the MRI • GCM-I. p is pressure. The lower surface is a coordinate surface. Above the level $p=p_I$, σ-surface coincides with a pressure surface.

From (0.1), the individual time derivative of pressure, ω, is given by

$$\omega\equiv\frac{dp}{dt}=\pi\dot{\sigma}+\sigma\left(\frac{\partial\pi}{\partial t}+\mathbf{v}\cdot\nabla\pi\right) \qquad (0.3)$$

where $\dot{\sigma}=d\sigma/dt$, $\mathbf{v}$ is the horizontal velocity and ∇ is the horizontal gradient operator. From (0.2), $\omega=\pi\dot{\sigma}$ for $\sigma<0$.

In the σ-coordinate, the earth's surface is a coordinate surface as well as a material surface. The kinematical boundary condition at the earth's surface is therefore

$$\dot{\sigma}=0 \quad \text{at} \quad \sigma=1 \qquad (0.4)$$

At the top of the model, we assume that no air parcels cross the top boundary. Thus,

$$\omega=\pi\dot{\sigma}=0 \quad \text{at} \quad \sigma=-1. \qquad (0.5)$$

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0.2 The equation of state

We consider that the model atmosphere is assumed to be a perfect gas.
Thus

$$\alpha = RT/p \tag{0.6}$$

where α is the specific volume, T is the temperature, and R is the gas constant. We do not distinguish R from that of dry air except when we consider buoyancy fluxes due to sub-grid scale turbulence and cumulus convection.

0.3 The hydrostatic equation

The hydrostatic equation in p-coordinate, $\frac{\partial \phi}{\partial p} = -\alpha$, is written as

$$\delta\phi = -\pi\alpha\delta\sigma \tag{0.7}$$

with use of the identity $\delta p = \pi\delta\sigma$, where δ is the differential under constant horizontal coordinate and time. ϕ is the geopotential $(=gz)$, g is the acceleration of gravity and z is height.

The form (0.7) may be transformed into the following equivalent forms;

$$\delta\phi = -RT\delta \ln p \tag{0.8}$$

$$= -c_p\theta\delta(p/p_0)^{\kappa} \tag{0.9}$$

$$= c_p \frac{d\ln\theta}{d\left(\frac{1}{\theta}\right)} \delta(p/p_0)^{\kappa} \tag{0.10}$$

$$\delta(\phi\sigma) = -(\pi\sigma\alpha - \phi)\delta\sigma \tag{0.11}$$

c_p is the specific heat at constant pressure, $\kappa = R/c_p$ and θ is the potential temperature, $T(p/p_0)^{-\kappa}$, where p_0 is a standard pressure.

0.4 Equation of continuity

The equation of continuity in p-coordinate is

$$\nabla_p \cdot \mathbf{v} + \frac{\partial \omega}{\partial p} = 0 \tag{0.12}$$

where ∇_p means ∇ operation on the constant p-surface. ∇_p is related to ∇_σ as

$$\nabla_p = \nabla_\sigma + \nabla_p \sigma \frac{\partial}{\partial \sigma} = \nabla_\sigma - \frac{\sigma}{\pi}\nabla\pi\frac{\partial}{\partial\sigma} \tag{0.13}$$

With the use of (0.13) and (0.3), (0.12) reduces to

$$(\nabla_\sigma \cdot \mathbf{v} - \frac{\sigma}{\pi} \nabla \pi \cdot \frac{\partial \mathbf{v}}{\partial \sigma}) + \frac{1}{\pi} \frac{\partial}{\partial \sigma} [\pi \dot{\sigma} + \sigma (\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla) \pi] = 0$$

and finally to

$$\frac{\partial \pi}{\partial t} + \nabla_\sigma \cdot (\pi \mathbf{v}) + \frac{\partial}{\partial \sigma} (\pi \dot{\sigma}) = 0 \tag{0.14}$$

Integrating (0.14) with respect to σ from $\sigma = -1$ to σ, we obtain

$$\int_{-1}^{\sigma} \frac{\partial \pi}{\partial t} d\sigma + \pi \dot{\sigma} = -\int_{-1}^{\sigma} \nabla \cdot (\pi \mathbf{v}) d\sigma \tag{0.15}$$

With the use of the definition of π, (0.15) gives

$$\pi \dot{\sigma} = -\int_{-1}^{\sigma} \nabla \cdot (\pi \mathbf{v}) d\sigma \quad \text{for } \sigma < 0 \tag{0.16}$$

$$\sigma \frac{\partial p_s}{\partial t} + \pi \dot{\sigma} = -\int_{-1}^{\sigma} \nabla \cdot (\pi \mathbf{v}) d\sigma \quad \text{for } \sigma > 0 \tag{0.17}$$

From (0.17), we obtain

$$\frac{\partial p_s}{\partial t} = -\int_{-1}^{1} \nabla \cdot (\pi \mathbf{v}) d\sigma \tag{0.18}$$

as $\dot{\sigma} = 0$ at $\sigma = 1$.

0.5 Momentum equation

The pressure gradient force in p-coordinate, $-\nabla_p \phi$, is transformed into

$$-\nabla_p \phi = -\nabla_\sigma \phi + \frac{\sigma}{\pi} \nabla \pi \frac{\partial \phi}{\partial \sigma} = -\nabla_\sigma \phi - \sigma \alpha \nabla \pi \tag{0.19}$$

where use has been made of (0.13) and (0.7). Then the horizontal component of the equation of motion becomes

$$\frac{d\mathbf{v}}{dt} + f\mathbf{k} \times \mathbf{v} + \nabla_\sigma \phi + \sigma \alpha \nabla \pi = \mathbf{F} \tag{0.20}$$

where

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_\sigma + \mathbf{v} \cdot \nabla_\sigma + \dot{\sigma} \frac{\partial}{\partial \sigma} \tag{0.21}$$

f is the vertical component of Coriolis vector ($= 2\Omega \sin\varphi$, Ω is the angular velocity of the earth, φ is latitude), **k** is the vertical unit vector, and **F** is the frictional force. Multiplying π and **v** to (0.20) and (0.14) respectively and adding them, we obtain the flux form of the equation of motion

$$\left(\frac{\partial}{\partial t}\right)_\sigma(\pi\mathbf{v})+\nabla_\sigma\cdot(\pi\mathbf{v})+\frac{\partial}{\partial\sigma}(\pi\dot{\sigma}\mathbf{v})+f\mathbf{k}\times\pi\mathbf{v}+\nabla_\sigma\cdot(\pi\phi)+(\pi\sigma\alpha-\phi)\nabla\pi$$

$$=\pi\mathbf{F} \quad (0.22)$$

or with the help of (0.11)

$$\left(\frac{\partial}{\partial t}\right)_\sigma(\pi\mathbf{v})+\nabla_\sigma\cdot(\pi\mathbf{v})+\frac{\partial}{\partial\sigma}(\pi\dot{\sigma}\mathbf{v})+f\mathbf{k}\times\pi\mathbf{v}+\nabla_\sigma(\pi\phi)-\frac{\partial(\phi\sigma)}{\partial\sigma}\nabla\pi$$

$$=\pi\mathbf{F} \quad (0.23)$$

0.6 Thermodynamic equation

The first law of thermodynamics is written as

$$c_p\frac{dT}{dt}=\omega\alpha+Q \quad (0.24)$$

where Q is the heating rate per unit mass. Multiplying π and c_pT to (0.24) and (0.14) respectively and adding them, the flux form

$$\frac{\partial}{\partial t}(\pi c_pT)+\nabla_\sigma\cdot(\pi\mathbf{v}c_pT)+\frac{\partial}{\partial\sigma}(\pi\dot{\sigma}c_pT)=\pi(\omega\alpha+Q) \quad (0.25)$$

is derived. Substitution of (0.3) into (0.25) gives us the form,

$$\frac{\partial}{\partial t}(\pi c_pT)+\nabla_\sigma\cdot(\pi\mathbf{v}c_pT)+(p/p_0)^\kappa\frac{\partial}{\partial\sigma}(\pi\dot{\sigma}c_p\theta)$$

$$=\pi c_pT\frac{\partial\ln(p/p_0)^\kappa}{\partial\pi}\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\pi+\pi Q \quad (0.26)$$

0.7 The continuity equation of water vapor and ozone

Let q be the mixing ratio of either water vapor or ozone. The continuity equation of q is written by

$$\frac{dq}{dt}=S \quad (0.27)$$

where S is the source of q. The flux form of q is

$$\left(\frac{\partial}{\partial t}\right)_\sigma(\pi q)+\nabla_\sigma\cdot(\pi\mathbf{v}q)+\frac{\partial}{\partial\sigma}(\pi\dot{\sigma}q)=\pi S \quad (0.28)$$