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A Theory on the Transparency of Sea Water

by

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Abstract

The present author derived theoretically the equation for the transparency of water as follows,

$$h=\frac{1}{2k(\lambda)}\log_e\frac{\left(1-\rho_0(\lambda)\right)^2\cdot\left(\frac{\rho(\lambda)}{\pi}-\frac{3}{16\pi}\frac{\alpha(\lambda)}{k(\lambda)}\right)}{\left\{\frac{3}{16\pi}\cdot\frac{\alpha(\lambda)}{k(\lambda)}\cdot\left(1-\rho_0(\lambda)\right)^2+\rho_0(\lambda)\right\}C(\lambda)},$$

and by the above equation discussed the relation between transparency and the oceanographical elements which have effects on it.

1. Theoretical consideration

Let us suppose a sea which has an infinite depth. Suppose that the solar radiation $J_0(\lambda)$ falls into water, then the light reflected back from the water surface is given by

$$B_w(\lambda)=\left[\left\{\frac{\sin(\varphi-\chi)}{\sin(\varphi+\chi)}\right\}^2+\left\{\frac{\tan(\varphi-\chi)}{\tan(\varphi+\chi)}\right\}^2\right]J_0(\lambda)\tag{1}$$

$$=\rho_0(\lambda)J_0(\lambda),$$

where λ, φ and χ are the wave length of light, the incident and refractive angles respectively. The intensity of light penetrating into water is respresented by

$$I_0(\lambda)=J_0(\lambda)-B_w(\lambda)=(1-\rho_0(\lambda))J_0(\lambda).\tag{2}$$

The vertical intensity of scattered light at the depth z is given by $\{3\alpha(\lambda)/8\pi\}\cdot I_0(\lambda)\cdot e^{-k(\lambda)z}$, where $\alpha(\lambda)$ and $k(\lambda)$ are respectively the scattering and extinction coefficient of water [1].

When we observe the scattered light through the water column the length of which is z, the light is exposed to extinction by the column and to reflection by the water surface. So on the surface we can see the scattered light at depth z as

$$\{3\alpha(\lambda)/8\pi\}\{1-\rho_0(\lambda)\}^2J_0(\lambda)e^{-2k(\lambda)z}.$$

Now, we must integrate the above quantity from 0 to z, since we measure actually the integrated intensity of light from all parts in the water column. Thus the intensity of scattered light from the water column is expressed as follows,

$$\{3\alpha(\lambda)/8\pi\}\{1-\rho_0(\lambda)\}^2 J_0(\lambda)\int_0^z e^{-2k(\lambda)z}\,dz$$
$$=\{3\alpha(\lambda)/16\pi k(\lambda)\}\{1-\rho_0(\lambda)\}^2 J_0(\lambda)(1-e^{-2k(\lambda)z}).$$

Next, the intensity of light reflected from the surface of the Secchi disc at the depth z is given by $\{\rho(\lambda)/\pi\}I_0(\lambda)e^{-k(\lambda)z}$ where $\rho(\lambda)$ is the albedo of the disc, and the disc is regarded as perfectly rough. We observe this reflected light as $\{\rho(\lambda)/\pi\}\{1-\rho_0(\lambda)\}^2 J_0(\lambda)e^{-2k(\lambda)z}$. Thus, the apparent intensity of light coming from the disc in water is expressed by the next formula;

$$B(\lambda)=B_w(\lambda)+\frac{3\alpha(\lambda)}{16\pi k(\lambda)}\{1-\rho_0(\lambda)\}^2 J_0(\lambda)(1-e^{-2k(\lambda)z}) + \frac{\rho(\lambda)}{\pi}\{1-\rho_0(\lambda)\}^2 J_0(\lambda)e^{-2k(\lambda)z}. \tag{3}$$

The scattering light from the water around the disc is given by

$$A(\lambda)=B_w(\lambda)+\frac{3\alpha(\lambda)}{8\pi}\{1-\rho_0(\lambda)\}^2 J_0(\lambda)\int_0^\infty e^{-2k(\lambda)z}\,dz = B_w(\lambda)+\frac{3\alpha(\lambda)}{16\pi k(\lambda)}\{1-\rho_0(\lambda)\}^2 J_0(\lambda). \tag{4}$$

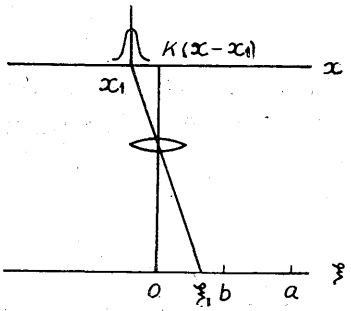

Fig. 1. The pattern of the image of Secchi disc under water upon the retina.

Now, if an observer looks upon the disc in water, the intensity distribution of light on his retina is expressed by $K(x-\xi_1)$, where ξ and x axes are the sea surface and the retina respectively [2], and the function $K(x)$ will depend on the character of his eyes as shown in Fig. 1. Now, the radii of the Secchi disc and the visual field are ob and oa respectively, then the image of the point ξ_1 on the retina is expressed by $K(x-x_1)$ and $x_1=\xi_1$.

So we recognize the disc as

$$\int_0^{2\pi}\int_\lambda\int_0^b e(\lambda)K(x-\xi)B(\lambda)\xi\,d\xi\,d\lambda\,d\theta,$$

where $e(\lambda)$ is the sensitivity of eyes. Thus the transparency of water can be calculated by the next equation

$$\frac{\int_0^{2\pi}\int_\lambda\left\{\int_0^b e(\lambda)K(x-\xi)B(\lambda)\xi\,d\xi-\int_b^a e(\lambda)K(x-\xi)A(\lambda)\xi\,d\xi\right\}d\lambda\,d\theta}{\int_0^{2\pi}\int_\lambda\int_b^a e(\lambda)K(x-\xi)A(\lambda)\xi\,d\xi\,d\lambda\,d\theta}=C, \tag{5}$$ [3]

where the constant C depends on the sensitivity of the observer's eyes and also the sea surface conditions. But then, we know little about $K(x-\xi)$, so we put approximately the above equation (5) as follows,

$$\frac{B(\lambda)-A(\lambda)}{A(\lambda)}=C(\lambda)\,. \tag{6}$$

Putting the equations (3) and (4) into (6), we get

$$e^{-2k(\lambda)z}=\frac{\left\{\frac{3}{16\pi}\frac{\alpha(\lambda)}{k(\lambda)}\{1-\rho_0(\lambda)\}^2+\rho_0(\lambda)\right\}C(\lambda)}{\{1-\rho_0(\lambda)\}^2\left(\frac{\rho(\lambda)}{\pi}-\frac{3}{16\pi}\frac{\alpha(\lambda)}{k(\lambda)}\right)}.$$

So the transparency of water can be expressed by the following formula;

$$h=\frac{1}{2k(\lambda)}\log_e\frac{\{1-\rho_0(\lambda)\}^2\left(\frac{\rho(\lambda)}{\pi}-\frac{3}{16\pi}\frac{\alpha(\lambda)}{k(\lambda)}\right)}{\left\{\frac{3}{16\pi}\frac{\alpha(\lambda)}{k(\lambda)}\{1-\rho_0(\lambda)\}^2+\rho_0(\lambda)\right\}C(\lambda)}. \tag{7}$$

If we take $\varphi=0$,

$$\rho_0(\lambda)=\left(\frac{n_{12}-1}{n_{12}+1}\right)^2=\left(\frac{n_2-n_1}{n_2+n_1}\right)^2,$$

which we may use approximately until $\varphi=30°$, where n is a refractive index.

Then,

$$h=\frac{1}{2k(\lambda)}\log_e\frac{\left\{\frac{4n}{(n+1)^2}\right\}^2\left(\frac{\rho(\lambda)}{\pi}-\frac{3}{16\pi}\frac{\alpha(\lambda)}{k(\lambda)}\right)}{\left[\frac{3}{16\pi}\frac{\alpha(\lambda)}{k(\lambda)}\left\{\frac{4n}{(n+1)^2}\right\}^2+\left(\frac{n-1}{n+1}\right)^2\right]C(\lambda)}. \tag{8}$$

2. Numerical calculation

The scattering coefficient of pure water is given by RAMAN, EINSTEIN and SMOLUKOWSKY [1] as follows

$$\alpha(\lambda)=\frac{\sigma RT}{N}(\mu^2-1)(\mu^2+1)\frac{1}{\lambda^4}\,.$$

Putting $\lambda=5200\,\mathring{A}$, $T=288\,°K$, $\sigma=49\times10^6$ C.G.S., $\mu=1{\cdot}34$, $R=22{\cdot}4\times10^3\,\mathrm{cm^3/mol}$, $N=6{\cdot}06\times10^{23}\,\mathrm{mol}^{-1}$, we get $\alpha=0.00223$. SAWYER [4] [5] got the value of $k=0{\cdot}019$ at $\lambda=5200\mathring{A}$ in distilled water. In natural water, VERCELLI [5] obtained the values $\alpha=0{\cdot}002$, $k=0{\cdot}06$, $0{\cdot}11$ at $\lambda=5300\mathring{A}$ and $5720\mathring{A}$. The refractive index is given as $n=1{\cdot}33524$ at $\lambda=5269{\cdot}978\mathring{A}$, $t=20°C$, so we get the values $\rho_0=0{\cdot}02$.

Assuming that the albedo of the white Secchi disc is 0.6 and the value of C is 1/100, we shall get the value of transparency for pure water, $h=173\,m$.

In Sargasso Sea, the value of the extinction coefficient $k=0.050$ was reported by G. L. CLARKE [5]. Using these quantities, we get $h=67.2\,m$. In fact, the transparency in Sargasso Sea was observed as $h=66.5\,m$. So we can say that the above calculation agrees well with the observation despite a simple assumption that the sea is homogeneous, i.e., k=const. and α=const. In other words, in Sargasso sea the water is quite transparent and homogeneous.

Now as seen in the equation (7), transparency is governed by $k(\lambda)$ and $\alpha(\lambda)$. So we can represent the nature of transparency as a function of $\alpha(\lambda)/k(\lambda)$. For, if we assume approximately that extinction is the sum of absorption and scattering, and that the absorption is equal to that in pure water, we get the equation $\alpha(\lambda)/k(\lambda)=1-0.019/k(\lambda)$ [6], which varies from zero to one as water becomes turbid. Next, for the value of $C(\lambda)$, different estimations are made by different authors, so here we take it to be 1/100 and 1/57 [1]. In actual observation, the ratio of light reflected back from the water surface such as 5% [1] is somewhat larger than the calculated one. We calculated hk by the equation (7). The results are given in the follwing table and Fig. 2.

The Table of hk

ρ/π \ α/k	0	0.02	0.2	0.4	0.6	0.8	1.0	Remarks
0.20	3.4	3.4	3.2	3.0	2.8	2.7	2.6	$C=\frac{1}{100}$, $\rho_0=0.02$
0.15	3.3	3.3	3.1	2.8	2.6	2.4	2.3	
0.20	3.1	3.1	2.9	2.7	2.5	2.3	2.2	$C=\frac{1}{57}$, $\rho_0=0.02$
0.15	3.0	3.0	2.8	2.5	2.3	2.1	2.0	
0.15	2.6	2.6	2.4	2.2	2.0	1.9	1.8	$C=\frac{1}{57}$, $\rho_0=0.05$

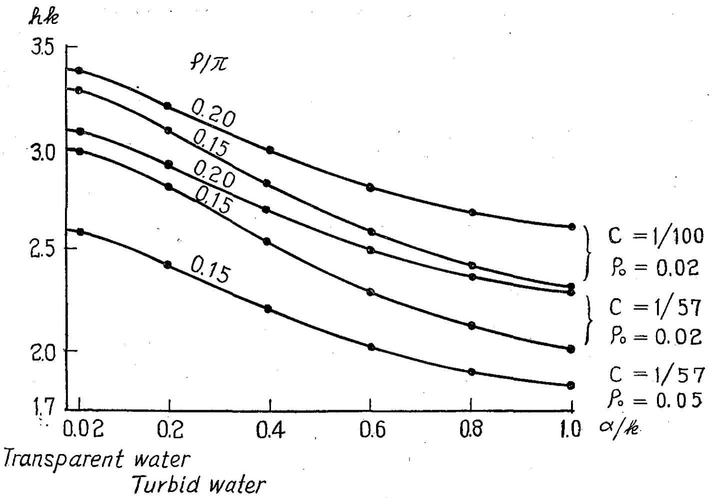

Fig. 2. Relation between (transparency × extinction coefficient) and (scatting coefficient/extinction coefficient)

We know as C and ρ_0 becomes larger and as ρ becomes smaller, the values of hk become smaller, and if ρ/π and ρ_0 are constant, the curve expressing hk gradually goes downwards keeping its form as C becomes larger. So if the albedo of the Secchi disc and the sea surface are constant, the relation between hk and α/k, or k is constant by any observer by shifting the coordinate axis, but if the albedo of the disc or the sea surface is not constant, the above relation of hk and α/k is different even by the same observer, i.e., in the case when C is constant. And as water becomes more turbid, i.e., α/k becomes closer to 1, the value of hk becomes smaller. Formerly, a formula such as $h=1.7/k$ or $hk=\text{const}$ [7] was deduced. If it is true, the value of C will be smaller, as the water becomes more turbid, i.e., it will become difficult to recognize the difference of brightness between the disc and its circumference, that is, our power of recognition of the difference of brightness will be lowered than in transparent water. Consequently transparency depends much on the difference of colour sensation as TAKENOUTI [1] says. And we should take secondary effects into consideration as water becomes more turbid, which generate the various colours of sea.

In conclusion, for the observation of the transparency of water, we must conduct further researches on $\alpha(\lambda)/k(\lambda)$, which plays a great role in the transparency of water, and we must be careful about the sea surface conditions which influence value of ρ_0 and C, and also we must keep the albedo of the disc ρ constant.

3. Correction to the transparency when a wire inclines

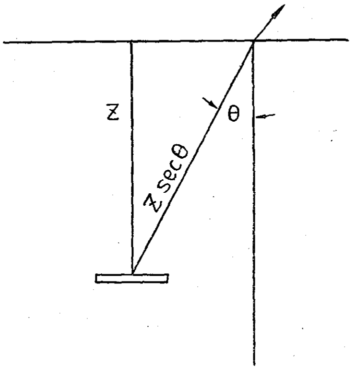

Fig. 3. Situation of a Secchi disc when a wire inclines.

When a wire suspending the Secchi disc has an inclination θ to the sea surface, and the disc keeps its plane horizontal as shown in Fig. 3, then, we are obliged to see the disc in an oblique direction; the intensity of scattered light from the water column is given by

$$\int_0^z \frac{3\alpha(\lambda)}{16\pi} J_0(\lambda)\{1-\rho_0(\lambda)\}e^{-k(\lambda)z}(1+\cos^2\theta)e^{-k(\lambda)\frac{z}{\cos\theta}}\{1-\rho_0'(\lambda)\}dz$$

$$=\frac{3}{16\pi}\frac{\alpha(\lambda)}{k(\lambda)}\frac{1+\cos^2\theta}{1+\frac{1}{\cos\theta}}\{1-\rho_0(\lambda)\}\{1-\rho_0'(\lambda)\}J_0(\lambda)\left(1-e^{-k(\lambda)\left(1+\frac{1}{\cos\theta}\right)z}\right).$$

When the angle θ is smaller than 30°, $\rho_0(\lambda)=\rho_0'(\lambda)=\{(n_{12}-1)/(n_{12}+1)\}^2$ and as θ is larger than 30°, the above equation holds approximately. The intensity of light reflected by the disc is given by

$$\frac{\rho(\lambda)}{\pi}\cos\theta\{1-\rho_0(\lambda)\}^2 J_0(\lambda)e^{-k(\lambda)\left(1+\frac{1}{\cos\theta}\right)z}.$$

So, we have the next equation for the apparent intensity of light coming from the disc in water instead of the equation (3),

$$(9)\quad B(\lambda)=B_w(\lambda)\cos\theta+\frac{3}{16\pi}\frac{\alpha(\lambda)}{k(\lambda)}\frac{1+\cos^2\theta}{1+\frac{1}{\cos\theta}}\{1-\rho_0(\lambda)\}^2 J_0(\lambda)\left(1-e^{-k(\lambda)\left(1+\frac{1}{\cos\theta}\right)z}\right)$$

$$+\frac{\rho(\lambda)}{\pi}\cdot\cos\theta\{1-\rho_0(\lambda)\}^2 J_0(\lambda)e^{-k(\lambda)\left(1+\frac{1}{\cos\theta}\right)z}.$$

The scattering light from the water around the disc is also given by

$$(10)\qquad A(\lambda)=B_w(\lambda)\cos\theta+\frac{3}{16\pi}\frac{\alpha(\lambda)}{k(\lambda)}\frac{1+\cos^2\theta}{1+\dfrac{1}{\cos\theta}}\{1-\rho_0(\lambda)\}^2J_0(\lambda).$$

So, putting equations (9) and (10) into (6), we get the next formula instead of equations (7) and (8):

$$(11)\qquad z=\frac{\cos\theta}{k(\lambda)(1+\cos\theta)}\log_e\frac{\{1-\rho_0(\lambda)\}^2\left(\dfrac{\rho(\lambda)}{\pi}-\dfrac{3}{16\pi}\dfrac{\alpha(\lambda)}{k(\lambda)}\dfrac{1+\cos^2\theta}{1+\cos\theta}\right)}{\left(\dfrac{3}{16\pi}\dfrac{\alpha(\lambda)}{k(\lambda)}\{1-\rho_0(\lambda)\}^2\dfrac{1+\cos^2\theta}{1+\cos\theta}+\rho_0(\lambda)\right)C(\lambda)}.$$

Then we have

$$z\doteqdot\frac{\cos\theta}{k(\lambda)(1+\cos\theta)}\log_e\frac{\{1-\rho_0(\lambda)\}^2\left(\dfrac{\rho(\lambda)}{\pi}-\dfrac{3}{16\pi}\dfrac{\alpha(\lambda)}{k(\lambda)}\right)}{\left(\dfrac{3}{16\pi}\dfrac{\alpha(\lambda)}{k(\lambda)}\{1-\rho_0(\lambda)\}^2+\rho_0(\lambda)\right)C(\lambda)}$$

$$=\frac{2\cos\theta}{1+\cos\theta}\cdot h.$$

So we have

$$h\doteqdot\frac{1+\cos\theta}{2\cos\theta}z=\frac{1+\cos\theta}{2}h_\theta,$$

where h_θ and h are respectively the length of the wire in water and the transparency of water. We have Fig. 4 which shows the increase in the length of the wire when it inclines in water.

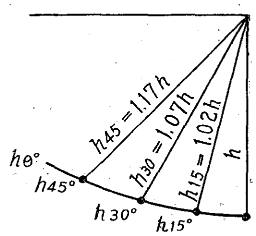

Fig. 4. Relation between apparent and true transparency when a wire inclines.

4. Seasonal variation

Assuming $\alpha(\lambda)=k(\lambda)-\text{const.}(\lambda)$, we have $\alpha(\lambda)/k(\lambda)=1-\text{const.}(\lambda)/k(\lambda)$. When the transparency varies periodically, the extinction coefficient also changes with the same period from the equation (7). Assuming that the colour of the sea is caused by the scattering and absorption of light, we can expect also that the colour of the sea will change annually with the same period of the extinction coefficient, or the variation of transparency. We can see the fact in actual observations, for example, in those reported by KOIZUMI [8].

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