where (4*b*) is the equation from which ψ is to be determined and (6*b*) is the one from which the corresponding pressure distribution is to be determined.

References

SATO, T., 1951: On the Horizontal Motion of the Atmosphere, Part 1, Stationary Motion, Papers in Meteorology and Geophysics, **2**, p. 343.

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On the Horizontal Motion of the Atomosphere (Ⅲ)

——Part 3. Vorticity Integral in the Non-Stationary Motion——

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Abstract

Here we derived a vorticity integral from the equations of motion under the following three assumptions:

(A) motion is horizontal,

(B) fluid is frictionless,

(C) fluid is autobarotropic.

Adopting the above assumptions, it is known from the equation of continuity that momentum can be expressed by applying stream-surface functions. Thus, we can obtain a vorticity integral from the equations of motion. The integral expresses a conservation law in a world of space and time, but not a conservation law in the ordinary space. We treated only the case in Cartesian coordinates.

Here we adopt the following three assumptions:

(A) motion is horizontal,

(B) fluid is frictionless,

(C) fluid is autobarotropic,

in integrating the euqations of motion. Giving attention to a vector (q, qu, qv) in the (t, x, y) space, it is known from the equation of continuity that the divergence of this vector is everywhere zero in that space, that is,

(1) $$\frac{\partial q}{\partial t}+\frac{\partial}{\partial x}(qu)+\frac{\partial}{\partial y}(qv)=0,$$

or

(1′) $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{1}{s}\Big(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\Big)s.$$

Accordingly, applying stream-surface functions $\alpha(t, x, y)$ and $\beta(t, x, y)$, the vector (q, qu, qv) can be expressed as

(2) $$(q, qu, qv)=\nabla\alpha\times\nabla\beta,$$

where ∇ is with respect to t, x and y, or in components:

(2a) $$q=\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial x}\equiv\frac{\partial(\alpha, \beta)}{\partial(x, y)},$$

(2b) $$qu=\frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial t}-\frac{\partial\alpha}{\partial t}\frac{\partial\beta}{\partial y}\equiv\frac{\partial(\alpha, \beta)}{\partial(y, t)},$$

(2c) $$qv=\frac{\partial\alpha}{\partial t}\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial t}\equiv\frac{\partial(\alpha, \beta)}{\partial(t, x)}.$$

The geometrical meaning of these functions (α, β) was given in the previous paper [1].

If the state of motion is stationary, then, putting $\beta=t$, we can regard t to be constant. Further, writing ψ instead of α, we get from (2b) and (2c) respectively

(3) $$qu=\frac{\partial\psi}{\partial y}, \qquad qv=-\frac{\partial\psi}{\partial x},$$

where ψ denotes the stream-line function for a two-dimensional vector (qu, qv). As this is the case when a β-plane coincides with the xy-plane, therefore, there appear in the xy-plane only the lines, on which the surfaces $\psi=\text{const}$ intersect the xy-plane, that is, the stream-lines in the xy-plane. Thus, this is the case which we treated in Part 1 [2].

From the assumptions (A) and (B) it is known that the equations of horizontal motion are expressed as

(4a) $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-2\,\omega\sin\theta\cdot v+s\frac{\partial p}{\partial x}=0,$$

(4b) $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+2\,\omega\sin\theta\cdot u+s\frac{\partial p}{\partial y}=0,$$

where the x-axis is directed eastward and the y-axis northward as in Part 1.

Now we calculate $\frac{\partial}{\partial x}(4b)-\frac{\partial}{\partial y}(4a)$, then, from the assumption (C),

$$\Big(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)\Big(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\Big)+2\,\omega\sin\theta\cdot\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)=0.$$

However, as

$$\Big(\frac{\partial}{\partial t}+u\frac{\partial}{\partial y}+v\frac{\partial}{\partial y}\Big)\Big\{s\Big(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\Big)\Big\}=s\Big(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)\Big(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\Big)$$

from (1′), therefore

$$\Big(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\Big)\Big\{s\Big(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\Big)+2\,\omega\sin\theta\Big\}=0.$$

Now, applying the stream-surface functions $(\alpha,\ \beta)$ in this equation, we get

$$\left\{\frac{\partial(\alpha,\beta)}{\partial(x,y)}\frac{\partial}{\partial t}+\frac{\partial(\alpha,\beta)}{\partial(y,t)}\frac{\partial}{\partial x}+\frac{\partial(\alpha,\beta)}{\partial(t,x)}\frac{\partial}{\partial y}\right\}$$

$$\times\left\{s\left[\frac{\partial}{\partial x}\left(s\frac{\partial(\alpha,\beta)}{\partial(t,x)}\right)-\frac{\partial}{\partial y}\left(s\frac{\partial(\alpha,\beta)}{\partial(y,t)}\right)+2\,\omega\sin\theta\right]\right\}=0.$$

Putting

$$s\left[\frac{\partial}{\partial x}\left(s\frac{\partial(\alpha,\beta)}{\partial(x,t)}\right)+\frac{\partial}{\partial y}\left(s\frac{\partial(\alpha,\beta)}{\partial(y,t)}\right)-2\,\omega\sin\theta\right]=-\Psi,$$

we get

$$\frac{\partial(\Psi,\alpha,\beta)}{\partial(t,x,y)}=0,$$

which is the necessary and sufficient condition for Ψ to be expressed as a function of α and β only. Thus we obtain an integral from the initial equations of motion (4a) and (4b)

$$(5)\qquad s\left\{\frac{\partial}{\partial x}\left(s\frac{\partial(\alpha,\beta)}{\partial(x,t)}\right)+\frac{\partial}{\partial y}\left(s\frac{\partial(\alpha,\beta)}{\partial(y,t)}\right)-2\,\omega\sin\theta\right\}=-\Psi(\alpha,\beta),$$

where $\Psi(\alpha,\beta)$ is an arbitrary function of α and β only. As α and β contain t, x and y, therefore Ψ depends upon t, x and y implicitely through α and β.

The absolute vorticity is expressed as

$$(6)\quad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+2\,\omega\sin\theta=-\left\{\frac{\partial}{\partial x}\left(s\frac{\partial(\alpha,\beta)}{\partial(x,t)}\right)+\frac{\partial}{\partial y}\left(s\frac{\partial(\alpha,\beta)}{\partial(y,t)}\right)-2\,\omega\sin\theta\right\},$$

so it will be seen that the equation (5) is an integral with respect to the absolute vorticity. The absolute vorticity multiplied with specific volume depends generally upon time, that is, the law of conservation of the absolute vorticity (multiplied with specific volume) in the ordinary sense does not hold under the assumptions (A), (B) and (C). However, as the functions α and β define the stream-tubes of the motion in the (t,x,y)-space, therefore it can be said that the equation (5) expresses the law of conservation of the absolute vorticity (multiplied with specific volume) along a stream-tube (α,β) in the world of space and time $(t,\ x,\ y)$.

Putting $\beta=t\,(=\text{const})$ and writing ψ instead of α in (5), we get

$$(7)\qquad s\left\{\frac{\partial}{\partial x}\left(s\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(s\frac{\partial\psi}{\partial y}\right)-2\,\omega\sin\theta\right\}=-\Psi(\psi),$$

which expresses the law of conservation of the absolute vorticity (multiplied with specific volume) in the stationary motion and coincides with that derived in Part 1.

References

[1] Sato, T., 1951: On the Motion Uniform in a Direction, Papers in Meteorology and Geophysics **2**, p. 339

[2] Sato, T., 1951: On the Horizontal Motion of the Atmosphere, Part 1, Stationary Motion, Papers in Meteorology and Geophysics **2**, p. 343