SHORTER CONTRIBUTIONS

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On the Horizontal Motion of the Atmosphere (II)

—Part 2. Stationary Motion in Spherical Coordinates—

by

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Abstract

Here we derived the integrals from the equations of horizontal motion in the case of spherical coordinates under the same assumptions as in Part 1. The "horizontal motion" which is called here means of course the motion on a spherical surface. The derived integrals express the same law of conservation of vorticity and energy respectively as in Part 1.

The assumptions adopted here are the four;

(A) motion is horizontal: $v_r=0$,

(B) fluid is frictionless,

(C) fluid is autobarotropic: $s=s(p)$,

(D) the state is stationary.

We adopt the spherical coordinates, the origin of which is put at the center of the earth, and adopt the usual symbols, where θ is the colatitude and the velocity is the linear one. The assumption (A) means $v_r=0$, that is, we are now considering the motion restricted on a spherical surface.

According to (A) and (D), the equation of continuity becomes

$$\frac{\partial}{\partial\theta}\left(r\sin\theta\cdot\frac{v_\theta}{s}\right)+\frac{\partial}{\partial\lambda}\left(r\cdot\frac{v_\lambda}{s}\right)=0, \tag{1}$$

or

$$\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\lambda}{\partial\lambda}+\frac{\cot\theta}{r}v_\theta=\frac{1}{s}\left(\frac{v_\theta}{r}\frac{\partial s}{\partial\theta}+\frac{v_\lambda}{r\sin\theta}\frac{\partial s}{\partial\lambda}\right). \tag{1'}$$

From (1) it is known that v_θ and v_λ can be expressed applying the stream-line function ψ for the horizontal momentum $\left(\frac{v_\theta}{s},\ \frac{v_\lambda}{s}\right)$, that is

$$v_\theta=-\frac{s}{r\sin\theta}\frac{\partial\psi}{\partial\lambda},\qquad v_\lambda=\frac{s}{r}\frac{\partial\psi}{\partial\theta}. \tag{2}$$

The equations of horizontal motion in spherical coordinates become according to (A), (B) and (D):

$$\text{(3a)}\qquad \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\lambda}{r\sin\theta}\frac{\partial v_\theta}{\partial\lambda}-\frac{\cot\theta}{r}v_\lambda{}^2-2\omega\cos\theta\cdot v_\lambda+\frac{s}{r}\frac{\partial p}{\partial\theta}=0,$$

$$\text{(3b)}\qquad \frac{v_\theta}{r}\frac{\partial v_\lambda}{\partial\theta}+\frac{v_\lambda}{r\sin\theta}\frac{\partial v_\lambda}{\partial\lambda}+\frac{\cot\theta}{r}v_\theta v_\lambda+2\omega\cos\theta\cdot v_\theta+\frac{s}{r\sin\theta}\frac{\partial p}{\partial\lambda}=0.$$

In order to derive a vorticity integral, we calculate $\frac{\partial}{\partial\lambda}(3a)-\frac{\partial}{\partial\theta}\{(3b)\times\sin\theta\}$, then

$$\left[\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+\frac{v_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}+\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\theta\sin\theta)+\frac{\partial v_\lambda}{\partial\lambda}\right\}\right]$$
$$\times\left[\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\lambda\sin\theta)-\frac{\partial v_\theta}{\partial\lambda}\right\}+2\omega\cos\theta\right]=0,$$

where

$$\zeta=(\text{rot}\,\boldsymbol{v})_r=\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\lambda\sin\theta)-\frac{\partial v_\theta}{\partial\lambda}\right\},$$

and

$$\text{div}_h\,\boldsymbol{v}=\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\theta\sin\theta)+\frac{\partial v_\lambda}{\partial\lambda}\right\},$$

whereas from (1′)

$$\left(\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+\frac{v_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}\right)\left\{s\left[\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\lambda\sin\theta)-\frac{\partial v_\theta}{\partial\lambda}\right\}+2\omega\cos\theta\right]\right\}$$
$$=s\left[\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+\frac{v_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}+\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\theta\sin\theta)+\frac{\partial v_\lambda}{\partial\lambda}\right\}\right]$$
$$\times\left[\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\lambda\sin\theta)-\frac{\partial v_\theta}{\partial\lambda}\right\}+2\omega\cos\theta\right].$$

Putting (2) into this we get

$$\left(\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial\lambda}-\frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\theta}\right)\left\{\frac{s}{r^2}\frac{\partial}{\partial\theta}\left(s\frac{\partial\psi}{\partial\theta}\right)+\frac{s}{r^2\sin^2\theta}\frac{\partial}{\partial\lambda}\left(s\frac{\partial\psi}{\partial\lambda}\right)+\frac{s^2\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}+2\omega s\cos\theta\right\}=0,$$

from which the following integral is obtained:

$$\text{(4)}\qquad \frac{s}{r^2}\frac{\partial}{\partial\theta}\left(s\frac{\partial\psi}{\partial\theta}\right)+\frac{s}{r^2\sin^2\theta}\frac{\partial}{\partial\lambda}\left(s\frac{\partial\psi}{\partial\lambda}\right)+\frac{s^2\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}+2\omega s\cos\theta=\Psi'(\psi).$$

As is seen from the expansion of absolute vorticity:

$$\text{(5)}\qquad \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\lambda\sin\theta)-\frac{\partial v_\theta}{\partial\lambda}\right\}+2\omega\cos\theta$$
$$=\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(s\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\lambda}\left(s\frac{\partial\psi}{\partial\lambda}\right)+\frac{s\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}+2\omega\cos\theta,$$

the equation (4) indicates that the absolute vorticity multiplied with specific volume is conserved along a stream-line in a horizontal (spherical) plane (the law of consrevation of absolute vorticity). It differs from the law on a plane surface only in the point that the effect of curvature enters in it.

Next, calculating $(3a)\times rd\theta+(3b)\times r\sin\theta d\lambda$, we get

$$v_\theta\left(\frac{\partial v_\theta}{\partial\theta}d\theta+\frac{\partial v_\theta}{\partial\lambda}d\lambda\right)+v_\lambda\left(\frac{\partial v_\lambda}{\partial\theta}d\theta+\frac{\partial v_\lambda}{\partial\lambda}d\lambda\right)+s\left(\frac{\partial p}{\partial\theta}d\theta+\frac{\partial p}{\partial\lambda}d\lambda\right)$$

$$+\left\{\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(v_\lambda\sin\theta)-\frac{\partial v_\theta}{\partial\lambda}\right]+2\omega\cos\theta\right\}\cdot r(v_\theta\sin\theta d\lambda-v_\lambda d\theta)=0,$$

whereas

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(v_\lambda\sin\theta)-\frac{\partial v_\theta}{\partial\lambda}\right]+2\omega\cos\theta=\frac{\Psi'(\psi)}{s},$$

and

$$r(v_\theta\sin\theta\, d\lambda-v_\lambda\, d\theta)=-s\,d\psi,$$

therefore

$$v_\theta\, dv_\theta+v_\lambda\, dv_\lambda-\Psi'(\psi)d\psi+s\,dp=0,$$

or

$$\left(\frac{s}{r}\frac{\partial\psi}{\partial\theta}\right)d\left(\frac{s}{r}\frac{\partial\psi}{\partial\theta}\right)+\left(\frac{s}{r\sin\theta}\frac{\partial\psi}{\partial\lambda}\right)d\left(\frac{s}{r\sin\theta}\frac{\partial\psi}{\partial\lambda}\right)+s\,dp=\Psi'(\psi)d\psi.$$

Integrating this, we get

$$\frac{1}{2}\left\{\frac{s^2}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2+\frac{s^2}{r^2\sin^2\theta}\left(\frac{\partial\psi}{\partial\lambda}\right)^2\right\}+\int s\,dp=\Psi(\psi), \tag{6}$$

or

$$\frac{1}{2}(v_\theta^2+v_\lambda^2)+\int s\,dp=\Psi. \tag{6'}$$

The equation (6) or (6′) indicates that energy is conserved along a stream-line on a spherical surface (the law of conservation of energy). It must be noticed that the effect of gravitation does not enter in it.

The independent variables are θ and λ, and the dependent ones are s (or p) and ψ in our problem, so that the variables s (or p) and ψ are to be determined from the integrals (4) and (6), provided suitable boundary conditions be given.

In the horizontal motion of the atmosphere, the variation of density is usually very small. If we assume s to be a function of θ only, then the above integrals become:

$$\frac{s^2}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{s^2}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\lambda^2}+\left(\frac{s}{r^2}\frac{\partial s}{\partial\theta}+\frac{s^2\cot\theta}{r^2}\right)\frac{\partial\psi}{\partial\theta}+2\omega s\cos\theta=\Psi'(\psi), \tag{4a}$$

$$\frac{1}{2}\left\{\frac{s^2}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2+\frac{s^2}{r^2\sin^2\theta}\left(\frac{\partial\psi}{\partial\lambda}\right)^2\right\}+\int s\,dp=\Psi(\psi). \tag{6a}$$

If we assume the specific volume to be constant, then we get:

$$\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\lambda^2}+\frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}+2\omega\cos\theta=\Psi'(\psi), \tag{4b}$$

$$\frac{1}{2}\left\{\frac{1}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial\psi}{\partial\lambda}\right)^2\right\}+p=\Psi(\psi), \tag{6b}$$

where (4b) is the equation from which ψ is to be determined and (6b) is the one from which the corresponding pressure distribution is to be determined.

References

SATO, T., 1951: On the Horizontal Motion of the Atmosphere, Part 1, Stationary Motion, Papers in Meteorology and Geophysics, 2, p. 343.

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On the Horizontal Motion of the Atomosphere (Ⅲ)

——Part 3. Vorticity Integral in the Non-Stationary Motion——

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Abstract

Here we derived a vorticity integral from the equations of motion under the following three assumptions:

(A) motion is horizontal,

(B) fluid is frictionless,

(C) fluid is autobarotropic.

Adopting the above assumptions, it is known from the equation of continuity that momentum can be expressed by applying stream-surface functions. Thus, we can obtain a vorticity integral from the equations of motion. The integral expresses a conservation law in a world of space and time, but not a conservation law in the ordinary space. We treated only the case in Cartesian coordinates.

Here we adopt the following three assumptions:

(A) motion is horizontal,

(B) fluid is frictionless,

(C) fluid is autobarotropic,

in integrating the euqations of motion. Giving attention to a vector (q, qu, qv) in the (t, x, y) space, it is known from the equation of continuity that the divergence of this vector is everywhere zero in that space, that is,