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On the Horizontal Motion of the Atmosphere
Part 1. Stationary Motion

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Abstract

Adopting the following three assumptions;

(A) motion is horizontal,

(B) fluid is frictionless,

(C) fluid is autobarotropic,

we tried here to integrate the equations of motion. For the sake of simplicity, we deal first with the case;

(D) the state is stationary.

As, in this case, it is known from the equation of continuity that we can apply the stream-line function ψ for the horizontal momentum, so we can once integrate the equations of motion, in which ψ and the specific volume are the dependent variations, and derive two integrals expressing conservations laws, that is, the law of conservation of absolute vorticity and that of energy.

Here we treated only of the case in Cartesian coordinates.

In some cases vertical motions, which are generally very small, play an important part in the atmosphere, but in most cases the state of motion of the atmosphere can be explained under the assumption of horizontal motion. Therefore we discuss here the horizontal motion of the atmosphere.

In integrating the equations of motion, we adopt the following four assumptions:

(A) motion is horizontal: $w=0$,

(B) fluid is frictionless,

(C) fluid is autobarotropic: $s=s(p)$,

(D) the state is stationary.

Then the equation of continuity becomes, according to (D), as

$$\frac{\partial}{\partial x}\left(\frac{u}{s}\right)+\frac{\partial}{\partial y}\left(\frac{v}{s}\right)=0, \tag{1}$$

or

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{1}{s}\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)s. \tag{1'}$$

From (1) it is known that we can apply the stream-line function for the horizontal momentum $\left(\frac{u}{s}, \frac{v}{s}\right)$, that is

(2) $$u = s\frac{\partial\psi}{\partial y}, \qquad v = -s\frac{\partial\psi}{\partial x}.$$

The equations of motion become, under the assumptions (A), (B) and (D), as

(3 a) $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - 2\omega\sin\theta\cdot v + s\frac{\partial p}{\partial x} = 0,$$

(3 b) $$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + 2\omega\sin\theta\cdot u + s\frac{\partial p}{\partial y} = 0,$$

where the x-axis is directed eastward and the y-axis northward.

First we calculate $\frac{\partial}{\partial x}(3b) - \frac{\partial}{\partial y}(3a)$, then we get, according to (C)

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + 2\omega\sin\theta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{\partial}{\partial y}(2\omega\sin\theta) = 0,$$

where the latitude θ is considered to be a function of y.

Now as

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left\{s\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right\} = s\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

from (1), therefore

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left[s\left\{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + 2\omega\sin\theta\right\}\right] = 0,$$

or

$$\left(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right)\left[s\left\{\frac{\partial}{\partial x}\left(s\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(s\frac{\partial\psi}{\partial y}\right) - 2\omega\sin\theta\right\}\right] = 0.$$

From this we get the following integral

(4) $$s\left\{\frac{\partial}{\partial x}\left(s\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(s\frac{\partial\psi}{\partial y}\right) - 2\omega\sin\theta\right\} = -\Psi'(\psi),$$

where $\Psi'(\psi)$ is an arbitrary function of ψ only.

As the vertical component ζ of relative vorticity is

(5) $$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\left\{\frac{\partial}{\partial x}\left(s\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(s\frac{\partial\psi}{\partial y}\right)\right\},$$

therefore (4) can be expressed also

(4′) $$s(\zeta + 2\omega\sin\theta) = \Psi'.$$

Thus the equation (4) or (4′) means that *the absolute vorticity multiplied with the specific volume is conserved along every stream-line* under the assumptions (A), (B), (C) and (D) (*Law of conservation of absolute vorticity*).

Next we calculate (3a)$\times dx$+(3b)$\times dy$, then

$$u\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + v\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) - \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + 2\omega\sin\theta\right)(vdx - udy) + s\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy\right) = 0.$$

Now as $$vdx - udy = -sd\psi,$$

therefore $$\frac{1}{2}d(u^2 + v^2) + \Psi'd\psi + sdp = 0,$$

or

$$\frac{1}{2}d\left\{\left(s\frac{\partial\psi}{\partial x}\right)^2+\left(s\frac{\partial\psi}{\partial y}\right)^2\right\}+\Psi'd\psi+sdp=0.$$

Integrating this equation, we get

$$\frac{1}{2}\left\{\left(s\frac{\partial\psi}{\partial x}\right)^2+\left(s\frac{\partial\psi}{\partial y}\right)^2\right\}+\int sdp=-\Psi(\psi), \tag{6}$$

where the integration constant is included in Ψ. This equation can also be expressed as

$$\frac{1}{2}(u^2+v^2)+\int sdp=-\Psi. \tag{6'}$$

The equation (6) or (6′) means that *the energy is conserved along every stream-line* under the assumptions (A), (B), (C) and (D) (*Law of conservation of energy*). It must be noticed here that the potential energy of gravitation is not concerned with this energy equation. It is because we are now dealing only with horizontal motion.

As the independent variables x and y, and the dependent ones are s (or p) and ψ in our problem, so the motion is to be fully determined from the two integrals (4) and (6), provided suitable boundary conditions be given. Of these two integrals, (6) can be considered an ultimate integral, but as (4) is only an intermediate integral, we must integrate this equation once more in order to determine the motion completely.

The variation of density q (or specific volume s) is usually very small in a horizontal plane in the atmosphere. If we assume the specific volume to be constant, we can put $s=1$ without losing generality, and the above equations become:

$$\nabla^2\psi-2\omega\sin\theta=-\Psi', \tag{4a}$$

$$\frac{1}{2}(\nabla\psi)^2+p=-\Psi, \tag{6a}$$

of which, (4a) is the equation from which the stream-line function ψ is to be determined and (6a) the one from which the corresponding pressure distribution is to be determined.

We discussed before on the stationary motion uniform in a direction [1], [2], [3]. The present paper followed the same course as the previous ones.

References

[1] Sato, T. 1951: On the Equations of Stationary Motion, Papers in Meteorology and Geophysics, **2**, p. 52.

[2] Sato, T. 1951: On the Structure of Typhoon. (yet unpublished).

[3] Sato, T. 1951: Dynamics of the Jet Stream, Papers in Meteorology and Geophysics, **2**, p. 132.