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Bernoullian Surfaces in Meteorology

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Abstract

It is shown that, when a barotropic fluid has any steady, frictionless motion, the sum of the kinetic energy per unit mass, the geopotential and the barotropic pressure function has no variation along stream lines as well as the total vortex lins; hence the sum is constant upon each of a family of the Bernoullian surfaces containing the stream lines and the absolute vortex lines through the common points.

The equation of frictionless motion in the atmosphere is

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+2[\Omega\times\boldsymbol{v}]=-\alpha\nabla p-\nabla\phi, \tag{1}$$

where $\boldsymbol{v}$ is the velocity, Ω the angular velocity of the earth's rotation, p the pressure, α the specific volume, ϕ the geopotential and t the time. A simple vector operation shows that

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+2[\Omega\times\boldsymbol{v}]=(2\Omega+\nabla\times\boldsymbol{v})\times\boldsymbol{v}+\frac{1}{2}\nabla q^2,$$

where q denotes the resultant velocity, and $(2\Omega+\nabla\times\boldsymbol{v})$ the absolute vorticity whose components are $(f_1+\xi)$, $(f_2+\eta)$ and $(f_3+\zeta)$. Eq. (1) may thus be written

$$\frac{\partial \boldsymbol{v}}{\partial t}+(2\Omega+\nabla\times\boldsymbol{v})\times\boldsymbol{v}=-\nabla\psi, \tag{2}$$

where ψ is the BERNOULLI function, i, e,

$$\psi=\int^p\alpha dp+\phi+\frac{1}{2}q^2,$$

assuming the fluid is barotropic. The function $\int^p\alpha dp$ is called the barotropic pressure function.

In steady motion, the equation of motion can be written as:

$$(2\Omega+\nabla\times\boldsymbol{v})\times\boldsymbol{v}=-\nabla\psi, \tag{3}$$

or

$$\begin{cases}(f_2+\eta)w-(f_3+\zeta)v=-\dfrac{\partial\psi}{\partial x},\\[2mm](f_3+\zeta)u-(f_1+\xi)w=-\dfrac{\partial\psi}{\partial y},\\[2mm](f_1+\xi)v-(f_2+\eta)u=-\dfrac{\partial\psi}{\partial z}.\end{cases} \tag{3a}$$

The differential equations of the system of stream lines are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}=\frac{ds}{q},$$

where ds is the line element along a stream line. If we now multiply the equations (3 a) by $\frac{u}{q}=\frac{dx}{ds}$, $\frac{v}{q}=\frac{dy}{ds}$, $\frac{w}{q}=\frac{dz}{ds}$, in order, and add, we obtain a result which may be written

(4) $$\frac{d\psi}{ds}=0,$$

or $$\int^{p}\frac{dp}{\rho}+\phi+\frac{q^2}{2}=\text{const.} \qquad \text{along a stream line,}$$

which permits us the wellknown BERNOULLI's law of variation of pressure along a stream line, i. e., the BERNOULLI's function is constant along a particular stream line.

A line of absolute vortex line is defined to be a line drawn from point to point, so that its direction is everywhere that of the absolute vorticity of the fluid motion. The differential equations of the system of absolute vortex lines are

$$\frac{dx}{f_1+\xi}=\frac{dy}{f_2+\eta}=\frac{dz}{f_3+\zeta}=\frac{dS}{Q},$$

where dS is the line element along a absolute vortex line whose vorticity strength is Q. If we multiply the equations (3 a) by

$$\frac{f_1+\xi}{Q}=\frac{dx}{dS}, \qquad \frac{f_2+\eta}{Q}=\frac{dy}{dS}, \qquad \frac{f_3+\zeta}{Q}=\frac{dz}{dS},$$

in order, and add, we obtain a result which may be written

$$\frac{d\psi}{dS}=0,$$

or $$\int^{p}\frac{dp}{\rho}+\phi+\frac{q^2}{2}=\text{const.} \qquad \text{along a absolute vortex line.}$$

It may be stated as follows: *When a barotropic fluid has any steady, frictionless motion, the sum of the kinetic energy per unit mass, the geopotential and the barotropic pressure function has no variation along the stream lines as well as the absolute vortex lines.* Thus the expression of the type ψ is constant upon each of a family of the BERNOULLIan surfaces containing the stream lines and the absolute vortex lines through the common points.

Consideration of orders of magnitude permits us to say that the stream lines are nearly horizontal, the absolute vortex lines are also nearly horizontal, *not vertical*, and that, therefore, a family of the BERNOULLIan surfaces ψ=const. is nearly horizontal.