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Notes on the Influence of Motion on the Temperature Lapse Rate and on the Stability of Dry and Moist Air

by

M. Takeuchi

Meteorological Research Institute

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Abstract

Concerning the influence of motion of air on the temperature lapse rate, there have been known several relations, for example Margules' formula concerning vertical motion of air, Byers' formula for application to isentropic weight chart, and Neamtan's stability tendency, etc.

In this paper Margules' formula is derived in such a manner that it can be applied to motion in general, with sufficient approximation. Also, from Margules' equation Neamtan's stability tendency is derived. In the case of moist adiabatic change, it is shown that a formula analogous to Margules' can be derived.

Application to weight chart in the case of moist adiabatic change is discussed. Also some applications of Charney's equation are discussed.

1. Derivation of Margules' formula in the general motion

The well-known Margules' formula concerning the influence of vertical motion of air on the temperature lapse rate is generalized as follows for the air column between two isentropic surfaces. In the isentropic motion, difference of potential temperature between the upper and lower surfaces of this air columns, $\delta\theta$, is conservative. From this and mass conservation,

$$\frac{\frac{1}{\theta}\delta\theta}{\rho S\delta n}=\frac{\frac{1}{\theta}\frac{\partial\theta}{\partial n}\delta n}{\rho S\delta n}=\frac{\frac{1}{\theta}\frac{\partial\theta}{\partial n}}{\rho S}$$

is also conservative, where the notations are as usual, S being the cross section of this air column on the isentropic surfaces and δn the distance between these surfaces. J. Charney (1948) showed that in large scale motion $\frac{\partial\theta}{\partial n}$ is replaced by $\frac{\partial\theta}{\partial z}$ approximately.

$$\frac{1}{\theta}\frac{\partial\theta}{\partial z}=\frac{1}{T}(\Gamma-\alpha)$$

where Γ is the dry adiabatic lapse rate and α is the lapse rate. Therefore $\frac{1}{SP}(\Gamma-\alpha)$ is conservative for the air column between two isentropic surfaces in large scale motion. Hence,

$$(1)\qquad \frac{d}{dt}\left[\frac{1}{SP}(\Gamma-\alpha)\right]=0$$

In equation (1), if we put

$$\pi=\frac{gP}{R\theta}\Big/(\Gamma-\alpha)=-\frac{\partial P}{\partial\theta},$$

eqution (1) is expressed as follows,

$$\frac{1}{SP}(\Gamma-\alpha)=\frac{g}{S\pi R\theta}$$

$$\frac{d}{dt}\left[\frac{g}{S\pi R\theta}\right]=0$$

also,

$$\frac{d\theta}{dt}=0$$

$$\frac{d}{dt}[S\pi]=0$$

$$\frac{d\pi}{dt}=-\frac{\pi}{S}\frac{dS}{dt}$$

$$\frac{1}{S}\frac{dS}{dt}\doteqdot\mathrm{div}_H V,\quad [\text{approximately}]$$

where

$$\mathrm{div}_H V=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

Hence,

$$\frac{d\pi}{dt}=-\pi\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial\pi}{\partial t}=-\left[\frac{\partial(\pi u)}{\partial x}+\frac{\partial(\pi v)}{\partial y}\right]$$

This is the same formula that was derived by S. M. Neamtan, in another way, and the left side of this equation is called stability tendency by the same author.

As easily shown, Byers' equation,

$$\frac{D_\theta'}{P_\theta}=\frac{D_0'}{P_0}$$

is also derived from Margules' equation, if we put

$$\Delta P=D',\quad SD\rho g=\text{const.},\quad S=\frac{\text{const}}{D\rho g},\quad D\approx\Delta z,\quad T\approx T_m,$$

and thus it can be seen that it is applicable to motion in general with sufficient approximation, for nearly horizontal state of isentropic surfaces in large-scale motion.

2. Stability change due to moist adiabatic change

We consider an air column between two equi-equivalent potential temperature surfaces. If we assume moist adiabatic change, difference of equivalent potential temperature at the top and bottom of this column $\delta\theta_e$ is conservative. Therefore, in the same manner as in the last section, if we assume

$$\frac{\partial\theta_e}{\partial z} \gg \frac{\partial\theta_e}{\partial x} \sim \frac{\partial\theta_e}{\partial y},$$

$$\frac{d}{dt}\left[\frac{1}{\theta_e}\frac{\partial\theta_e}{\partial z}\frac{1}{S\rho}\right]=0$$

$$\frac{1}{\theta_e}\frac{\partial\theta_e}{\partial z}=\frac{1}{T}\frac{\partial T}{\partial z}-\frac{AR}{c_p}\frac{\partial(p-e)}{(p-e)\partial z}+\frac{\partial\left(\frac{\gamma\xi}{c_pT}\right)}{\partial z}$$

$$=\frac{1}{T}\left(\frac{\partial T}{\partial z}+\alpha_m\right)=\frac{1}{T}(\alpha_m-\alpha),$$

where e the vapour tension, ξ the mixing ratio, aud α_m the moist adiabatic lapse rate.

Hence,

$$\frac{d}{dt}\left[\frac{1}{sp}(\alpha_m-\alpha)\right]=0$$

This is Margules' formula in the cace of moist adiabatic change. In this case, α_m is not constant.

Thus,

$$\alpha_1=\alpha_0+(\alpha_{m0}-\alpha_0)\left(1-\frac{s_1p_1}{s_0p_0}\right)+\Delta\alpha_m$$

$$\Delta\alpha_m=\alpha_{m1}-\alpha_{m0}$$

where suffix 0 indicates the initial, and suffix 1 the final state.

If we put $D'=\Delta p$, where Δp is the pressure difference between two equi-equivalent potential temperature serfaces, and assume that equi-equivalent potential temperature surface are quasi-horizontal, an analogous equation due to H. Byers is derived easily,

$$\frac{D'_{\theta e}}{P_{\theta e}}=\frac{D_0'}{P_0}.$$

Therefore, if we consider the weight chart for the equi-equivalent potential temperature surfaces in place of isentropic surfaces, we see the relative stability in the case of moist adiabatic chane in different air masses. But, in this case, the change of moist adiabatic lapse rate must be considered. Of course, in lower troposphere, it is thought that moist weight chart, by which designate the above chart, is more usuful than the isentropic weight chart, though more labous and time are necessary to construct it.

3. Some Applications of Charney's equation.

If we substitute Helmholtz's vorticity theorem on the isentropic surface, $ZS=$ const., in Margules' equation, and elimlnate S, we get,

$$\frac{d}{dt}\left[\frac{z}{p}(\Gamma-\alpha)\right]=0$$

where Z is the vertical component of absolute vorticity.

This is a slightly modified form of Charney's equation.

$$\frac{d}{dt}[\rho^{-1}\nabla\theta(2\,\Omega+\nabla\times W)]=0$$

and recognized by Dr. H. Arakawa immediately from this equation, as a very usuful formula. Dr. Charney derived the above formula by combination of three conservative natures for an air column between two isentropic surfaces,

1. mass conservation,
2. conservation of potential temperature,
3. $ZS=$const.,

also it is known that the combination of 1. and 3. makes Rossby's equation,

$$\frac{Z}{\Delta p}=\text{const.},$$

and as mentioned above, it is shown in this paper that the combination of 1. and 2. becomes Margules' equation.

a) Large-scale horizontal motion

When we consider large-scale horizontal motion, where pressure is approximately unchanged, we can write from the above equation,

$$\alpha_1=\alpha_0+(\Gamma-\alpha_0)\left(1-\frac{z_0}{z_1}\right)$$

Now we assume quasi-constant relative vorticity compared with the change of Coriolis parameter, as for the ordinary state, $2\omega\sin\varphi\sim10^{-4}$ and relative vorticity $\zeta\sim 10^{-5}\sim10^{-6}$

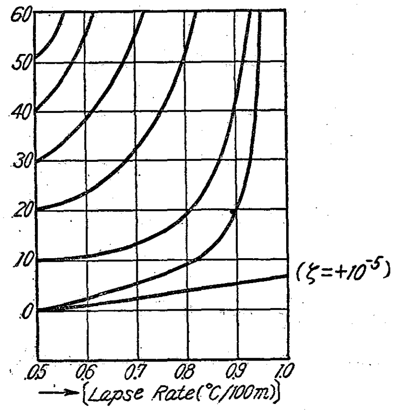

Fig. 1 Northward motion of stable air layer ($\zeta=-10^{-5}$)

In the case of originally stable stratification, $\Gamma>\alpha_0$, the lapse rate change as follows. When $z_1>z_0$, i.e. when the air moves northward, $\alpha_1>\alpha_0$, i.e. the lapse rate becomes larger, and when $z_1<z_0$, i.e. when the air moves southward, $\alpha_1<\alpha_0$, i.e. the lapse rate becomes smaller and when $z_1\leqq\left(\frac{\Gamma-\alpha_0}{\Gamma}\right)z_0$ a temperature inversion will occur. On the other hand, in the case of originally unstable stratification, $\Gamma<\alpha_0$, the effect of northward and southward motions is just the opposite. Southward motion increases the lapse rate: northward motion makes it smaller, and the lapse rate approaches the dry adiabatic lapse rate. But this effect is small in higher latitudes, as Figs 1 ~ 4 show.

In the lower atmosphere, modifications of air masses due to the earth's surface is very large, and also in the upper atomosphere in long continued motion, the effects of radiation and mixing of different air masses become very large.

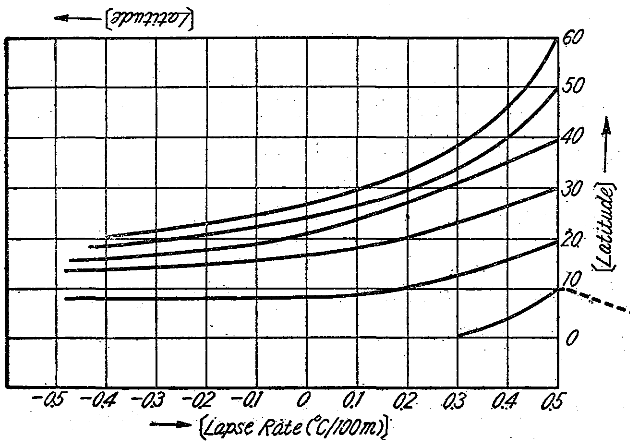

Fig. 2 Southward motion of stable air layer ($\zeta=-10^{-5}$)

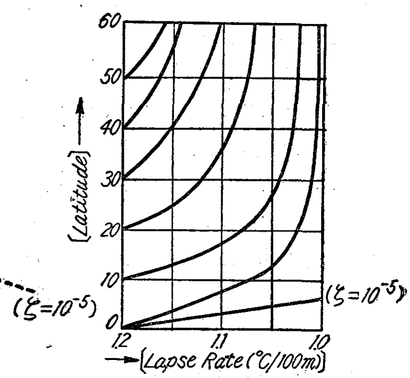

Fig. 3 Northward motion of unstable air layer ($\zeta=-10^{-5}$)

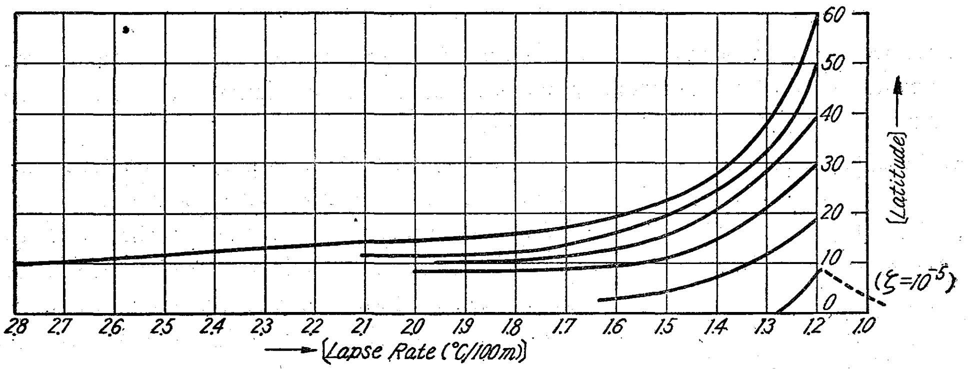

Fig. 4 Southward motion of unstable air layer ($\zeta=10^{-5}$)

Therefore, in these cases the assumption of adiabatic change will not be suitable, and the above-mentioned relations do not hold good.

b) Locally vertical motion.

When we consider the locally vertical motions the horizontal scale of which is not very large,

$$\alpha_1=\alpha_0+(\Gamma-\alpha_0)\left(1-\frac{P_1}{P_0}\right)$$

if we assume absolute vorticity is approximately unchanged, i. e. when change of relative vorticity is not unusually large, except the case in the lower latitudes.

This is discussed in ordinary textbooks, for example in B. Haurwitz's.

When $P_1 \geqq P_0\left(\frac{\Gamma}{\Gamma-\alpha_0}\right)$, in the case of $\Gamma > \alpha_0$, $P_1 > P_0$, i.e. descending motion of an ordinarily stable air column, subsidence inversion will occur.

c) Moist adiabatic motion.

As $\theta_e=\theta_e(P,T,\xi)$, on the equi-equivalent potential temperature surfaces ZS is not generally constant.

If we assume barotropic atmosphere or, on the equi-equivalent potential temperature surfaces, mixing ratio $\xi=$const , we get the following modified form of Charney's equation:

$$\frac{Z_0}{P_0}(\alpha_{m0}-\alpha_0)=\frac{Z_1}{P_1}(\alpha_{m1}-\alpha_1)$$

$$\alpha_1=\alpha_0+(\alpha_{m0}-\alpha_0)\left(1-\frac{Z_0 p_1}{Z_1 p_0}\right)+\Delta\alpha_m$$

$$\Delta\alpha_m=\alpha_{m1}-\alpha_{m0}$$

Except for correction of change of the moist adiabatic lapse rate, if we replace dry adabatic stability by moist adiabatic stability, conditions are quite analogous to the case of dry adiabatic change.

4. Conclusion.

In this paper it was shown that Margules' formula is more useful than it has been considered. For, in the first place, it can be used in general motions, and secondly an analogous formula can be derived in the case of moist adiabatic change. Also some applications of Charney's equation, were discussed.

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