

On the Equations of Stationary Motion

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Abstract

From the equations of motion including inertia terms, which are very difficult for integration, we can derive intermediate integrals in some cases, if we adopt the following assumptions:

- (A) motion is stationary,
- (B) motion is uniform in one direction,
- (C) fluid is auto-barotropic,

because, in this case, the equation of continuity becomes simple and the motion is expressed by applying a pseudo-streamline function. Here is stated the general process of integration, only in the case when Cartesian coordinates are used. The treatment of the case when curvilinear coordinates are used, will be reserved for another opportunity. Some of the applications of these integrals are already published by the author.

Here we will treat generally of the motion of frictional fluid moving under the influence of deflecting force on the rotating earth. We adopt here only the following three assumptions:

- (A) motion is stationary,
- (B) motion is uniform in one direction,
- (C) fluid is auto-barotropic.

We consider the coefficient of friction to be constant. Then the equation of motion, which is expressed by the generalized Navier-Stokes' equation, is written in vector form as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + 2\Omega \times \mathbf{v} + \nabla \phi + s \nabla p - \kappa \nabla^2 \mathbf{v} - \frac{\kappa}{3} \nabla \operatorname{div} \mathbf{v} = 0.$$

The equation of continuity is

$$\frac{\partial s}{\partial t} - s^2 \operatorname{div} \left(\frac{\mathbf{v}}{s} \right) = 0.$$

Now we simplify these equations by applying the above assumptions (A), (B) and (C). Thereby the origin of coordinates is placed on the earth's surface, the x -axis, which is considered to be the direction in which the motion is uniform, is directed to the north by angle β from the east, and the z -axis to vertically upward.

Then the equations of motion become:

$$(1) \quad v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - 2\omega \sin \theta \cdot v + 2\omega \cos \theta \cos \beta \cdot w = \kappa \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

$$(2) \quad v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + 2\omega \sin \theta \cdot u - 2\omega \cos \theta \sin \beta \cdot w + s \frac{\partial v}{\partial y} \\ = \kappa \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \frac{\kappa}{3} \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right),$$

$$(3) \quad v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} - 2\omega \cos \theta \cos \beta \cdot u + 2\omega \cos \theta \sin \beta \cdot v + g + s \frac{\partial w}{\partial z} \\ = \kappa \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{\kappa}{3} \frac{\partial}{\partial z} \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right).$$

The equation of continuity becomes

$$(4) \quad \frac{\partial}{\partial y} \left(\frac{v}{s} \right) + \frac{\partial}{\partial z} \left(\frac{w}{s} \right) = 0.$$

From eq. (4) it will be seen that the momentum $\left(\frac{v}{s}, \frac{w}{s} \right)$ is expressed by applying the streamline function ψ as

$$(5) \quad v = s \frac{\partial \psi}{\partial z}, \quad w = -s \frac{\partial \psi}{\partial y},$$

where ψ is the pseudo-streamline function for the velocity (v, w) .

Now we will rewrite the equations of motion under application of the pseudo-streamline function ψ .

First, putting eq. (5) into eq. (1) we get

$$s \frac{\partial \psi}{\partial z} \frac{\partial u}{\partial y} - s \frac{\partial \psi}{\partial y} \frac{\partial u}{\partial z} - 2\omega \sin \theta \cdot s \frac{\partial \psi}{\partial z} - 2\omega \cos \theta \cos \beta \cdot s \frac{\partial \psi}{\partial y} = \kappa \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

which will be rewritten as

$$\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} (u - 2\omega \sin \theta \cdot y + 2\omega \cos \theta \cos \beta \cdot z) - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} (u - 2\omega \sin \theta \cdot y + 2\omega \cos \theta \cos \beta \cdot z) \\ + \nu \nabla^2 (u - 2\omega \sin \theta \cdot y + 2\omega \cos \theta \cos \beta \cdot z) = 0,$$

where $\nu = \frac{\kappa}{s}$. This will further be rewritten as

$$(6) \quad \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} + \nu \nabla^2 \right) (u - 2\omega \sin \theta \cdot y + 2\omega \cos \theta \cos \beta \cdot z) = 0.$$

This is an equation with respect to momentum and can not be simply integrated in general. In special cases, however, we can derive from this an intermediate integral concerning momentum as will be seen next.

(i) If we assume the fluid to be frictionless, then putting $\nu = 0$, we get from eq. (6)

$$\left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} \right) (u - 2\omega \sin \theta \cdot y + 2\omega \cos \theta \cos \beta \cdot z) = 0,$$

from which an intermediate integral can be derived as follows,

$$(6a) \quad u - 2\omega \sin \theta \cdot y + 2\omega \cos \theta \cos \beta \cdot z = V(\psi),$$

where $V(\psi)$ is an arbitrary function of ψ .

Further, if the effect of deflecting force can be neglected, then we get

$$(6 a') \quad u = V(\psi).$$

This indicates that the velocity component u is constant along a streamline $\psi = \text{const.}$ in $y-z$ plane.

(ii) Even though we assume the specific volume (or density) of fluid be constant, eq. (6) can not be simplified, but if we neglect further the effect of deflecting force, then we get

$$(6 b') \quad \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} + \nu \nabla^2 \right) u = 0.$$

Next, calculating $-\frac{\partial}{\partial z}\{\text{eq. (2)}\} + \frac{\partial}{\partial y}\{\text{eq. (3)}\}$, we get

$$(a) \quad \begin{aligned} & \nu \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + w \frac{\partial}{\partial z} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \frac{\partial v}{\partial y} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \frac{\partial w}{\partial z} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \\ & - 2\omega \left(\cos \theta \cos \beta \frac{\partial u}{\partial y} + \sin \theta \frac{\partial u}{\partial z} \right) + 2\omega \cos \theta \sin \beta \left(\frac{\partial s}{\partial y} \frac{\partial \psi}{\partial z} - \frac{\partial s}{\partial z} \frac{\partial \psi}{\partial y} \right) \\ & - \kappa \nabla^2 \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = 0, \end{aligned}$$

whereas

$$\begin{aligned} & \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} \right) \left\{ s \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \right\} \\ & = s \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} \right) \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \left(\frac{\partial \psi}{\partial y} \frac{\partial s}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial s}{\partial y} \right) \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \\ & = - \left\{ \left(v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \right\}, \end{aligned}$$

further, as

$$\nabla^2 \left\{ s \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \right\} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \nabla^2 s + s \nabla^2 \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + 2 \nabla s \cdot \nabla \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right),$$

therefore

$$\kappa \nabla^2 \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = \left\{ \nu \nabla^2 - 2 \frac{\nu}{s} \nabla s \cdot \nabla + 2 \frac{\nu}{s^2} (\nabla s)^2 - \frac{\nu}{s} \nabla^2 s \right\} \left\{ s \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \right\}.$$

Further, as

$$s \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = s^2 \frac{\partial^2 \psi}{\partial y^2} + s^2 \frac{\partial^2 \psi}{\partial z^2} + s \frac{\partial s}{\partial y} \frac{\partial \psi}{\partial y} + s \frac{\partial s}{\partial z} \frac{\partial \psi}{\partial z},$$

therefore eq. (a) becomes

$$(7) \quad \begin{aligned} & \left\{ \frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} + \nu \nabla^2 - 2 \frac{\nu}{s} \nabla s \cdot \nabla + 2 \frac{\nu}{s^2} (\nabla s)^2 - \frac{\nu}{s} \nabla^2 s \right\} \left(s^2 \frac{\partial^2 \psi}{\partial y^2} + s^2 \frac{\partial^2 \psi}{\partial z^2} \right. \\ & \left. + s \frac{\partial s}{\partial y} \frac{\partial \psi}{\partial y} + s \frac{\partial s}{\partial z} \frac{\partial \psi}{\partial z} \right) - 2\omega \left\{ \cos \theta \cos \beta \frac{\partial u}{\partial y} + \sin \theta \frac{\partial u}{\partial z} - \cos \theta \sin \beta \left(\frac{\partial s}{\partial y} \frac{\partial \psi}{\partial z} - \frac{\partial s}{\partial z} \frac{\partial \psi}{\partial y} \right) \right\} \\ & = 0. \end{aligned}$$

This equation is with respect to vorticity and can not be simply integrated in general, except in the special cases where intermediate integrals can be derived.

(i) First, we consider that the fluid is frictionless. As we have already derived in this case an intermediate integral (6 a) from the equation concerning

momentum, we get now

$$\frac{\partial u}{\partial y} = V' \frac{\partial \psi}{\partial y} + 2\omega \sin \theta, \quad \frac{\partial u}{\partial z} = V' \frac{\partial \psi}{\partial z} - 2\omega \cos \theta \cos \beta.$$

Putting these into eq. (7), where $\nu=0$ is assumed, then

$$\begin{aligned} & -2\omega \left\{ \cos \theta \cos \beta \frac{\partial u}{\partial y} + \sin \theta \frac{\partial u}{\partial z} - \cos \theta \sin \beta \left(\frac{\partial s}{\partial y} \frac{\partial \psi}{\partial z} - \frac{\partial s}{\partial z} \frac{\partial \psi}{\partial y} \right) \right\} \\ & = -2\omega \left\{ \frac{\partial \psi}{\partial y} \left(\cos \theta \cos \beta \cdot V' + \cos \theta \sin \beta \frac{\partial s}{\partial z} \right) - \frac{\partial \psi}{\partial z} \left(-\sin \theta \cdot V' + \cos \theta \sin \beta \frac{\partial s}{\partial y} \right) \right\} \\ & = \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} \right) \{ V'(V + 2\omega \sin \theta \cdot y - 2\omega \cos \theta \cos \beta \cdot z) - 2\omega s \cos \theta \sin \beta \}. \end{aligned}$$

Therefore eq. (7) becomes

$$\begin{aligned} & \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} \right) \left\{ s^2 \frac{\partial^2 \psi}{\partial y^2} + s^2 \frac{\partial^2 \psi}{\partial z^2} + s \frac{\partial s}{\partial y} \frac{\partial \psi}{\partial y} + s \frac{\partial s}{\partial z} \frac{\partial \psi}{\partial z} \right. \\ & \left. + V'(V + 2\omega \sin \theta \cdot y - 2\omega \cos \theta \cos \beta \cdot z) - 2\omega s \cos \theta \sin \beta \right\} = 0, \end{aligned}$$

from which the following intermediate integral can be derived

$$(7a) \quad s^2 \frac{\partial^2 \psi}{\partial y^2} + s^2 \frac{\partial^2 \psi}{\partial z^2} + s \frac{\partial s}{\partial y} \frac{\partial \psi}{\partial y} + s \frac{\partial s}{\partial z} \frac{\partial \psi}{\partial z} + V'(V + 2\omega \sin \theta \cdot y - 2\omega \cos \theta \cos \beta \cdot z) - 2\omega s \cos \theta \sin \beta = \Psi'(\psi).$$

Further, if the effect of deflecting force can be neglected, then we get

$$(7a') \quad s^2 \frac{\partial^2 \psi}{\partial y^2} + s^2 \frac{\partial^2 \psi}{\partial z^2} + s \frac{\partial s}{\partial y} \frac{\partial \psi}{\partial y} + s \frac{\partial s}{\partial z} \frac{\partial \psi}{\partial z} = \Psi',$$

where only Ψ' is written instead of $\Psi' - V'V$, as Ψ' and V are both arbitrary functions of ψ only. Eq. (7a) or (7a') indicates that the x -component of vorticity is conserved along streamlines, when the effect of friction is negligible.

If the fluid can be considered to be incompressible, then we get

$$(7a'') \quad V^2 \psi + V'(V + 2\omega \sin \theta \cdot y - 2\omega \cos \theta \cos \beta \cdot z) - 2\omega \cos \theta \sin \beta = \Psi',$$

because, as we are now dealing with stationary motion, we can consider in this case that the fluid is homogeneous and incompressible and therefore we can put $s = \text{const} = 1$ without losing generality.

Further if the effect of deflecting force is negligible, then we get

$$(7a''') \quad \nabla^2 \psi = \Psi'.$$

(ii) Next we consider the fluid to be incompressible, then eq. (7) becomes

$$(7b) \quad \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} + \nu \nabla^2 \right) \nabla^2 \psi - 2\omega \left(\cos \theta \cos \beta \frac{\partial u}{\partial y} + \sin \theta \frac{\partial u}{\partial z} \right) = 0.$$

This equation cannot be simply integrated, as the velocity component u appears in it.

If, further, the effect of deflecting force is negligible, then we get

$$(7b') \quad \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial}{\partial y} + \nu \nabla^2 \right) \nabla^2 \psi = 0.$$

As only ψ appears as dependent variable in this equation, this can be integrated in special cases, and is often applied in dealing with the stationary motion of viscous fluid. We can refer the readers to the papers⁽¹⁾ by G. Hamel, C. W. Oseen, etc., on

this point.

Next, eq. (3) becomes

$$(b) \quad -s \left(\frac{\partial s}{\partial y} \frac{\partial \psi}{\partial y} + s \frac{\partial^2 \psi}{\partial y^2} \right) \frac{\partial \psi}{\partial z} + s \left(\frac{\partial s}{\partial z} \frac{\partial \psi}{\partial y} + s \frac{\partial^2 \psi}{\partial y \partial z} \right) \frac{\partial \psi}{\partial y} - 2\omega \cos \theta \cos \beta \cdot u \\ + 2\omega s \cos \theta \sin \beta \frac{\partial \psi}{\partial z} + g + s \frac{\partial p}{\partial z} - \kappa \nabla^2 w - \frac{\kappa}{3} \frac{\partial}{\partial z} \left(\frac{v}{s} \frac{\partial s}{\partial y} + \frac{w}{s} \frac{\partial s}{\partial z} \right) = 0,$$

whereas

$$-s \left(\frac{\partial s}{\partial y} \frac{\partial \psi}{\partial y} + s \frac{\partial^2 \psi}{\partial y^2} \right) \frac{\partial \psi}{\partial z} + s \left(\frac{\partial s}{\partial z} \frac{\partial \psi}{\partial y} + s \frac{\partial^2 \psi}{\partial y \partial z} \right) \frac{\partial \psi}{\partial y} \\ = \left[s \frac{\partial s}{\partial z} \left\{ \left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 \right\} + s^2 \left(\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial y \partial z} + \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial z^2} \right) \right] \\ - s \frac{\partial s}{\partial z} \left(\frac{\partial \psi}{\partial z} \right)^2 - s \frac{\partial s}{\partial y} \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial z} - s^2 \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial y^2} - s^2 \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial z^2} \\ = \frac{\partial}{\partial z} \left\{ \frac{1}{2} \left[s^2 \left(\frac{\partial \psi}{\partial y} \right)^2 + s^2 \left(\frac{\partial \psi}{\partial z} \right)^2 \right] \right\} - s \frac{\partial \psi}{\partial z} \left\{ s \nabla^2 \psi + \frac{\partial \psi}{\partial y} \frac{\partial s}{\partial y} + \frac{\partial \psi}{\partial z} \frac{\partial s}{\partial z} \right\}, \\ - \kappa \nabla^2 w = \kappa \{ s \nabla^2 + 2 \nabla s \cdot \nabla + \nabla^2 s \} \frac{\partial \psi}{\partial y},$$

and

$$- \frac{\kappa}{3} \frac{\partial}{\partial z} \left(\frac{v}{s} \frac{\partial s}{\partial y} + \frac{w}{s} \frac{\partial s}{\partial z} \right) = \frac{\kappa}{3} \frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial y} \frac{\partial s}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial s}{\partial y} \right),$$

therefore eq. (b) becomes

$$(8) \quad \frac{\partial}{\partial z} \left\{ \frac{1}{2} \left[s^2 \left(\frac{\partial \psi}{\partial y} \right)^2 + s^2 \left(\frac{\partial \psi}{\partial z} \right)^2 \right] + g z + \int s d p \right\} - \frac{\partial \psi}{\partial z} \left\{ s^2 \frac{\partial^2 \psi}{\partial y^2} + s^2 \frac{\partial^2 \psi}{\partial z^2} \right. \\ \left. + s \frac{\partial s}{\partial y} \frac{\partial \psi}{\partial y} + s \frac{\partial s}{\partial z} \frac{\partial \psi}{\partial z} - 2\omega s \cos \theta \sin \beta \right\} - 2\omega \cos \theta \cos \beta \cdot u + \kappa (s \nabla^2 + 2 \nabla s \cdot \nabla + \nabla^2 s) \frac{\partial \psi}{\partial y} \\ + \frac{\kappa}{3} \frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial y} \frac{\partial s}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial s}{\partial y} \right) = 0.$$

This is an equation with respect to energy and cannot be simply integrated in general.

(ii) We consider first the case without friction. As in this case we have already derived intermediate integrals (6 a) and (7 a) respectively from equations concerning momentum and vorticity, we can put these into eq. (8). Then we get

$$- \frac{\partial \psi}{\partial z} \left\{ s^2 \frac{\partial^2 \psi}{\partial y^2} + s^2 \frac{\partial^2 \psi}{\partial z^2} + s \frac{\partial s}{\partial y} \frac{\partial \psi}{\partial y} + s \frac{\partial s}{\partial z} \frac{\partial \psi}{\partial z} - 2\omega s \cos \theta \sin \beta \right\} - 2\omega \cos \theta \cos \beta \cdot u \\ = - \frac{\partial \psi}{\partial z} \{ \Psi' - V'(V + 2\omega \sin \theta \cdot y - 2\omega \cos \theta \cos \beta \cdot z) \} - 2\omega \cos \theta \cos \beta (V + 2\omega \sin \theta \cdot y \\ - 2\omega \cos \theta \cos \beta \cdot z) = - \Psi' \frac{\partial \psi}{\partial z} + \frac{1}{2} \{ (V + 2\omega \sin \theta \cdot y - 2\omega \cos \theta \cos \beta \cdot z)^2 \} \frac{\partial \psi}{\partial z},$$

therefore eq. (8) can be integrated as

$$(8a) \quad \frac{1}{2} \left\{ s^2 \left(\frac{\partial \psi}{\partial y} \right)^2 + s^2 \left(\frac{\partial \psi}{\partial z} \right)^2 + (V + 2\omega \sin \theta \cdot y - 2\omega \cos \theta \cos \beta \cdot z)^2 \right\} + g z + \int s d p = \Psi.$$

As Ψ is only a function ψ , this equation indicates that the energy is conserved

along streamlines, when the effect of friction is negligible.

If further the effect of deflecting force can be neglected, then we get

$$(8 a') \quad \frac{1}{2} \left\{ s^2 \left(\frac{\partial \psi}{\partial y} \right)^2 + s^2 \left(\frac{\partial \psi}{\partial z} \right)^2 + V^2 \right\} + gz + \int s dp = \Psi.$$

If the fluid can be considered to be incompressible, then we get

$$(8 a'') \quad \frac{1}{2} \left\{ \left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 + (V + 2\omega \sin \theta \cdot y - 2\omega \cos \theta \cos \beta \cdot z)^2 \right\} + gz + p = \Psi,$$

Further if the effect of deflecting force is negligible, then we get

$$(8 a''') \quad \frac{1}{2} \left\{ \left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 + V^2 \right\} + gz + p = \Psi.$$

(ii) If we assume the fluid to be incompressible, then eq. (8) becomes

$$(8 b) \quad \frac{\partial}{\partial z} \left\{ \frac{1}{2} \left[\left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 \right] + gz + p \right\} - \frac{\partial \psi}{\partial z} \{ \nabla^2 \psi - 2\omega \cos \theta \sin \beta \} \\ - 2\omega \cos \theta \cos \beta \cdot u + \kappa \nabla^2 \frac{\partial \psi}{\partial y} = 0.$$

Further, if the effect of deflecting force is negligible, then we get

$$(8 b') \quad \frac{\partial}{\partial z} \left\{ \frac{1}{2} \left[\left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 \right] + gz + p \right\} - \frac{\partial \psi}{\partial z} \nabla^2 \psi + \kappa \nabla^2 \frac{\partial \psi}{\partial y} = 0.$$

Now we have derived the equations which represent the motion under assumptions (A), (B) and (C). They are the equation (6) with respect to momentum, the equation (7) with respect to vorticity and the equation (8) with respect to energy.

These equations can not be easily integrated, but when the fluid is frictionless, they are all integrable. The intermediate integrals in this case are shown as (6 a), (7 a) and (8 a). These indicate respectively that the momentum (x -component of), the vorticity (x -component of) and the energy are conserved along streamlines, when the effect of friction is negligible. We discussed before atmospheric motions by applying these relations. ⁽¹⁾

Also as to frictional fluid, the relations representing motion become relatively simple, if we neglect the effect to deflecting force and consider the fluid to be incompressible. The equations in this case are shown as (6 b'), (7 b') and (8 b'). Of these, eq. (7 b') is the one from which ψ is to be determined. When ψ is determined from eq. (7 b'), then u is determined from (6 b') and p from (8 b'). These relations, especially eq. (7 b') are already widely applied in discussing the motion of viscous fluid.

References

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