# A Proposition on the Advection Problem

by

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**Abstract**

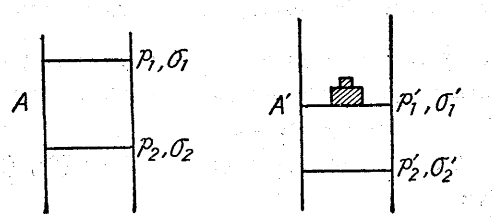
From the physical point of view, it is desirable to deal the advection problem with the conservative property $\frac{\partial \theta}{\partial p}$ and differential method. As a result a formulae convenient in estimating the horizontal divergence is obtained.

**Introduction**

The problem of advection has been discussed by many writers since 1926 (Exner)[1] and the formula concerning surfaee pressure change $\delta p$ and advective pressure change $\delta \pi$ has already been much improved upon. But, as was pointed out by T. Watanabe,[2] the vertical integration in those formulae should be done by the initial state instead of the final. The problem of advection is, in other words, that of vertical stratification. In order to avoid such confusions, it is desirable to deal with it by the aid of such quantities as are conserved throughout the process and, moreover, by differential relation which excludes intermediate state. Following the course of these ideas, in this paper, conservative quantity $\frac{\partial \sigma}{\partial p}$ is treated of in the first two sections and previous formulae are rewritten by the differential relations in the last section.

**1. Introduction of a more favourable quantity**

Let us consider the air column A to be changed into A′ by advection over it. This process is assumed to be polytropic change. Let us denote the pressure by p, the polytropic potential temperature by $\sigma$ and the upper and the lower part of the column by subscript 1 and 2. Then $\sigma_1=\sigma_1'$ and $\sigma_2=\sigma_2'$, and therefore $\sigma_1-\sigma_2=\sigma_1'-\sigma_2'$ or $\Delta\sigma=\Delta\sigma'$. Further, $\Delta p=\Delta p'$ if the horizontal divergence does not occur as it was assumed hitherto. It is found under these assumptions that the ratio $\frac{\Delta\sigma}{\Delta p}$ is conserved.




Let us denote

$$(1) \qquad \frac{\partial \sigma}{\partial p} \equiv \frac{\frac{\partial \sigma}{\partial z}}{\frac{\partial p}{\partial z}} = -\frac{\frac{\partial \sigma}{\partial z}}{g\rho},$$

and operate individual differentiation. We get

$$\frac{d}{dt}\left(\frac{\partial\sigma}{\partial p}\right)=-\frac{1}{g}\frac{d}{dt}\left(\frac{1}{\rho}\frac{\partial\sigma}{\partial z}\right)=-\frac{1}{g\rho}\left[\frac{d}{dt}\left(\frac{\partial\sigma}{\partial z}\right)-\frac{1}{\rho}\frac{d\rho}{dt}\frac{\partial\sigma}{\partial z}\right]$$
$$=-\frac{1}{g\rho}\left[\frac{\partial}{\partial z}\left(\frac{d\sigma}{dt}\right)-\frac{\partial u}{\partial z}\frac{\partial\sigma}{\partial x}-\frac{\partial v}{\partial z}\frac{\partial\sigma}{\partial y}-\frac{\partial w}{\partial z}\frac{\partial\sigma}{\partial z}+\frac{\partial\sigma}{\partial z}\mathrm{div}_2 V+\frac{\partial w}{\partial z}\frac{\partial\sigma}{\partial z}\right]$$
$$=-\frac{1}{g\rho}\left[\frac{\partial\sigma}{\partial z}\mathrm{div}_2 V-\frac{\partial u}{\partial z}\frac{\partial\sigma}{\partial x}-\frac{\partial v}{\partial z}\frac{\partial\sigma}{\partial y}\right]$$

from the assumption of polytropic change $\frac{d\sigma}{dt}=0$, and the eqation of continuity

$$\frac{1}{\rho}\frac{d\rho}{dt}+\mathrm{div}_2 V+\frac{\partial w}{\partial z}=0\ .$$

This relation may be written in the case of adiabatic change as

$$(2)\qquad \frac{d}{dt}\left(ln\frac{\partial\theta}{\partial p}\right)=\mathrm{div}_2 V-\frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial x}\right)_\theta-\frac{\partial v}{\partial z}\left(\frac{\partial z}{\partial y}\right)_\theta\ ,$$

where $\left(\frac{\partial z}{\partial x}\right)_\theta$ and $\left(\frac{\partial z}{\partial y}\right)_\theta$ indicate the inclination of isentropic surface.

The above relation shows that the individual change of the quantity $ln\frac{\partial\theta}{\partial p}$ is the summation of horizontal divergence and horizontal component of vector production between shear vector and normal of the isentropic surface. The latter term is zero when the shear vector is on the isentropic surface, and max. or min. when that is perpendicular to this, and shows the effect of advection in upper or lower layer. In other words, (1) expresses clearly that divergence has the effect of increasing stability; and wind shear acting so as to decrease the inclination of isentropic surface has the same effect and vice versa.

By the way, the same relation holds for the horizontal thermal stability. Namely,

$$(3)\qquad \frac{d}{dt}\left(ln\ \rho^{-1}\frac{\partial\theta}{\partial x}\right)=\mathrm{div}_{yz}V-\frac{\partial v}{\partial x}\left(\frac{\partial x}{\partial y}\right)_\theta-\frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial z}\right)_\theta$$

$$(4)\qquad \frac{d}{dt}\left(ln\ \rho^{-1}\frac{\partial\theta}{\partial y}\right)=\mathrm{div}_{zx}V-\frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial z}\right)_\theta-\frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial x}\right)_\theta$$

for adiabatic change. If we take the coordinate such as to make $\frac{\partial\theta}{\partial x}=0$, that is, $y$ axis coincides with the horizontal gradient of potential temperature, we may consider (4) (the last term is dropped) only. In this expression, $\mathrm{div}_{zx}V$ denotes two-dimensional divergence in $z$-$x$ plane and $\left(\frac{\partial y}{\partial z}\right)_\theta$, $\left(\frac{\partial y}{\partial x}\right)_\theta$ same as above. Eq. (4) shows that divergence in the vertical plane containing the iso-thermal line and horizontal wind shear acting to increase the inclination of isentropic surface have the effect of frontogenesis.

The thermodynamic equations (2), (3), (4) combined with vorticity equation give

$$(5)\qquad \frac{d}{dt}\left[\rho^{-1}\nabla\sigma\cdot(2\Omega+\nabla\times V)\right]=0\ ,$$

which shows the conservation of potential vorticity as given by Charney[3].

## 2. Practical Application

In order to apply the above formula to one-point observations, (2) may be rewritten as

$$(6)\qquad \frac{d'}{dt}\left(ln\frac{\partial\theta}{\partial p}\right)=\mathrm{div}_2 V+\rho^{-1}\left(u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}\right)-\left(\frac{\partial\vartheta}{\partial z}\right)^{-1}\frac{\partial}{\partial z}\left\{u\frac{\partial\theta}{\partial x}+v\frac{\partial\vartheta}{\partial y}\right\},$$

where $\frac{d'}{dt}=\frac{\partial}{\partial t}+w\frac{\partial}{\partial z}$. In this form, the second term on the right hand is usually negligible as compared with $\mathrm{div}_2 V$, and the third term is also negligible except in the special case where wind shear and inclination of isentropic surface are both large or the curvature of isentropic surface is large. Divergence of $10^{-5}$ is rather familiar but wind shear of $10^{-2}$ and inclination o' $10^{-3}$ are rare except in the frontal zone. Therefore, when considering on a large scale, which we are obliged to do so because available data are of 6 or 12 hours' interval*, we may regard

$$(7)\qquad \frac{d'}{dt}\left(ln\frac{\partial\theta}{\partial p}\right)=\mathrm{div}_2 V$$

instead of (6).

From one-point observation, we may calculate $\frac{\partial\theta}{\partial p}$, as a function of $\theta$. Taking $ln\frac{\partial\theta}{\partial p}$ for ordinate and $\theta$ for abscissa, we may estimate $\frac{d'}{dt}\left(ln\frac{\partial\theta}{\partial p}\right)$ approximately from the difference between the two successive curves.

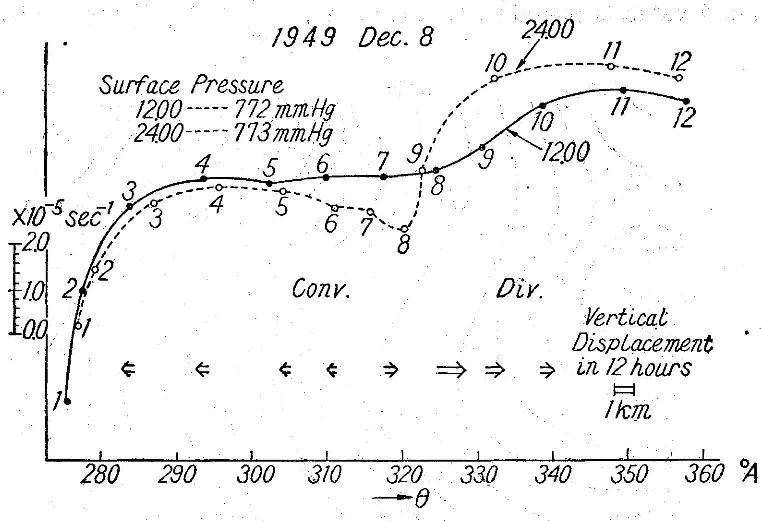



Fig. 1

* Normal horizontal and vertical velocity are of the order of 10 m/sec and 1 cm/sec respectively, therefore, spacial increment $\Delta x$, $\Delta z$, should be taken as 400 km and 400 m for time interval of 12 hours.

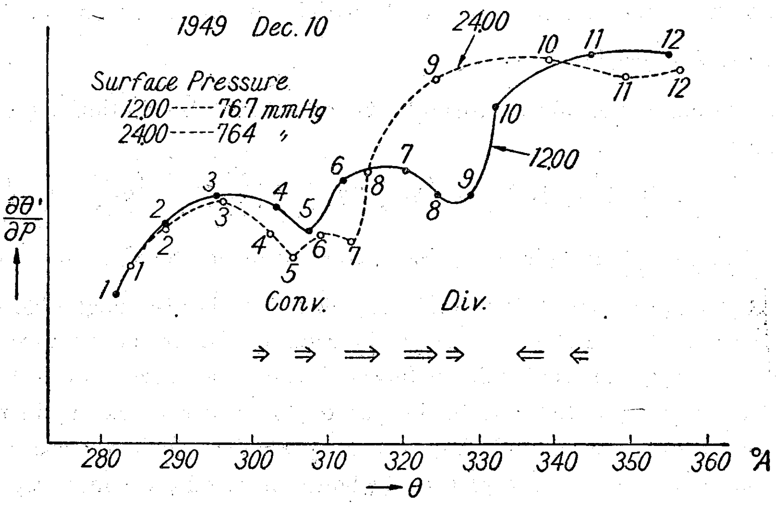



Fig. 2

Fig. 1 shows the vertical distributions of divergence and convergence on the neighborhood of anticyclone. Weather map is given in Fig. 3. Fig. 2 shows those in the neighborhood of cyclone. Weather map is given in Fig. 4.

It may be seen also from the arrows in Figs 1 & 2, which show the vertical displacement between 12 hrs' interval, that the approximate estimation of $\frac{d'}{dt}\left(ln\frac{\partial\theta}{\partial p}\right)$ by the same $\theta$ value is passable, that is to say, $u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}$ is negligible.

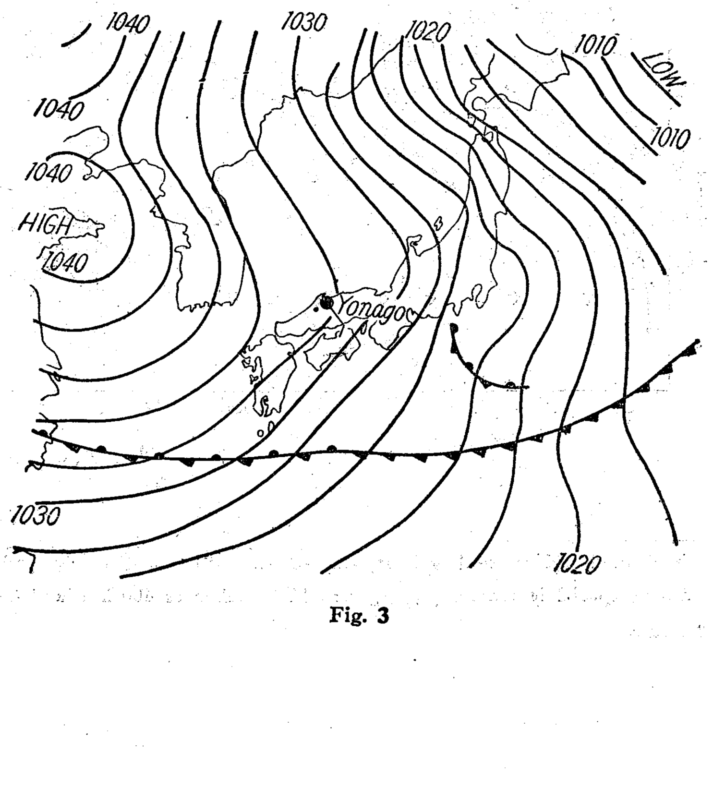



Fig. 3

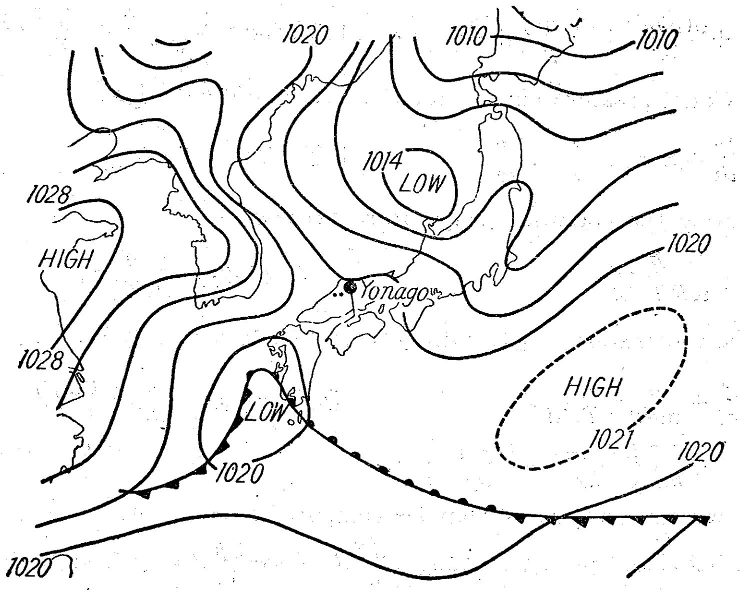



Fig. 4

The advantages of this method are the following,

1) $\frac{\partial\theta}{\partial p}$ may be calculated from direct observation data, and the error by height estimation may be avoided.

2) Not only the advection is investigated but also divergence can be estimated quantitatively.

### 3. Rewriting previous formulae into differential relation

Previous formulae of advection relate the advection function $\delta\pi$ with local pressure change $\delta p$ caused by advection. Local variation and individual variation between before and after the advection are denoted by the same notation $\delta$. These difference relations are not obtained without some assumptions concerning intermediate state. Mass additon or subtraction does not occur in the free atmosphere. These facts make this problem difficult. From this point of view, we intend to follow some formulae by differential relation in the continuous fluid and explain the advection functian $\delta\pi$ more physically.

In the first place, Ertel's formula in 1938[4] may be rewritten as follows.

Adiabatic equation is [Ertel's computation]

$$[1]\quad \frac{\partial\theta}{\partial t}+u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}+w\frac{\partial\theta}{\partial z}=0 . \qquad [1]'\quad \delta\theta+\frac{\partial\theta}{\partial x}\delta x+\frac{\partial\theta}{\partial y}\delta y+\frac{\partial\theta}{\partial z}\delta z=0 .$$

Equation of continuity is

$$[2]\quad \frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 . \quad [2]'\quad \delta\rho+\frac{\partial(\rho\delta x)}{\partial x}+\frac{\partial(\rho\delta y)}{\partial y}+\frac{\partial(\rho\delta z)}{\partial z}=0 .$$

Combining [1] and [2]

[3] $\frac{\partial(\rho\theta)}{\partial t}+\frac{\partial(\rho\theta u)}{\partial x}+\frac{\partial(\rho\theta v)}{\partial y}+\frac{\partial(\rho\theta w)}{\partial z}=0$. [3]′ $\delta(\rho\theta)+\frac{\partial(\rho\theta\delta x)}{\partial x}+\frac{\partial(\rho\theta\delta y)}{\partial y}+\frac{\partial(\rho\theta\delta z)}{\partial z}=0$.

From individual pressure change relation and static equation

[4] $\frac{\partial p}{\partial t}=\frac{dp}{dt}+g\rho w-u\frac{\partial p}{\partial x}-v\frac{\partial p}{\partial y}$. [4]′ $\delta p=\delta\pi+g\rho\delta z$.

Differentiating the potential temperature expression

[5] $\frac{\partial(\rho\theta)}{\partial t}=\frac{1}{æR}\frac{\theta}{T}\frac{\partial p}{\partial t}$. [5]′ $\delta(\rho\theta)=\frac{1}{æR}\frac{\theta}{T}\delta p$.

Putting the integration of [3] with respect to $z$ into $w$ in [4]

[6] $\frac{dp}{dt}=\frac{\partial p}{\partial t}+\frac{g}{æR\theta}\int_0^z\frac{\theta}{T}\frac{\partial p}{\partial t}dz+\frac{1}{\theta}\int_0^z\left\{\frac{\partial(\rho\theta u)}{\partial x}+\frac{\partial(\rho\theta v)}{\partial y}\right\}dz$.

[6]′ $\delta\pi=\delta p+\frac{g}{æR\theta}\int_0^z\frac{\theta}{T}\delta p dz+\frac{1}{\theta}\int_0^z\left\{\frac{\partial(\rho\theta\delta x)}{\partial x}+\frac{\partial(\rho\theta\delta y)}{\partial y}\right\}dz$.

Therefore we can understand that $\delta\pi$ computed from Ertel's formula should be regarded as individual pressure change $\frac{dp}{dt}dt$ since $\frac{\partial p}{\partial t}dt=\delta p$ from definition.

Instead of computing $w$ from the integration of [3], we may use the following relation

$$\frac{\partial}{\partial z}\left(\frac{dp}{dt}\right)=\frac{\partial u}{\partial z}\frac{\partial p}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial p}{\partial y}-\mathrm{div}_2\,V\frac{\partial p}{\partial z}. \tag{8}$$

From equation of continuity we get

$$\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{d\rho}{dt}-\mathrm{div}_2\,V=-\frac{1}{æp}\frac{dp}{dt}-\frac{1}{g\rho}\frac{\partial}{\partial z}\left(\frac{dp}{dt}\right)+\frac{\partial u}{\partial z}\frac{1}{g\rho}\frac{\partial p}{\partial z}+\frac{\partial v}{\partial z}\frac{1}{g\rho}\frac{\partial p}{\partial y}. \tag{9}$$

Integrating (9)

$$w=-\frac{1}{æ}\int_0^z\frac{1}{p}\frac{dp}{dt}dz-\frac{1}{g}\int_0^z\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{dp}{dt}\right)dz+\frac{1}{g}\int_0^z\left\{\frac{\partial u}{\partial z}\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial v}{\partial z}\frac{1}{\rho}\frac{\partial p}{\partial y}\right\}dz. \tag{10}$$

Putting (11) into [4] we get

$$\frac{\partial p}{\partial t}=\frac{dp}{dt}-\frac{g}{æR}\frac{p}{T}\int_0^z\frac{1}{p}\frac{dp}{dt}-\rho\int_0^z\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{dp}{dt}\right)dz. \tag{11}$$

Eq. (10) has the same form as the formula given by H. L. Kuo[5] and later by M. Yoshitake[6].

### *References*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