ROSSBY-HAURWITZ's Equation for the Conservation of Vertical Vorticity Component in the Baroclinic Atmosphere

by

H. Arakawa

Meteorological Research Institute

(Received September 11, 1950)

CHARNEY's equation of the conservation of vorticity is given by

$$(1)\qquad \frac{d}{dt}\left\{\rho^{-1}\cdot\nabla\theta\cdot(2\mathbf{W}+\mathrm{rot}\,V)\right\}=0,$$

where t is the time, $2\mathbf{W}$ the earth's vorticity, V the velocity of air and θ a conservation quantity in the atmosphere depending only on the pressure p and the density ρ:

$$(2)\qquad \frac{d\theta}{dt}=0, \qquad \text{or} \qquad \theta=const.$$

If isentropic motion is assumed so that θ may be the potential temperature. It is noteworthy that the only assumptions made in deriving the conservation equation (1) are that the motion is isentropic and frictionless, and the conservation equation (1) holds good *even in the baroclinic atmosphere.*

Since the isentropic surfaces are quasi-horizontal in the large-scale systems, Eq. (1) may be written as:

$$(3)\qquad \frac{d}{dt}\left\{\rho^{-1}\frac{\partial\theta}{\partial z}(f+\zeta)\right\}=0,$$

where z is the vertical axis, f the Coriolis parameter, and ζ the relative vertical vorticity component.

From the definition of potential temperature, it is seen that

$$(4)\qquad \frac{1}{\theta}\frac{\partial\theta}{\partial z}=\frac{1}{T}\left(\frac{\partial T}{\partial z}+\Gamma\right),$$

where T is the air temperature in absolute scale, and Γ the dry adiabatic lapse rate (See, Brunt, D., 1939: *Physical and Dynamical Meteorology,* p.41, Eq. 39). Let a thin layer of air be raised adiabatically from a level where the pressure is p to a level where the pressure is p', where the column of air changes its horizontal extent by spreading outward or by convergence. Let the horizontal area is initially S and changes to S' in the new position. For a given mass of air $S\cdot\Delta p$ will remain constant, or

$$(5)\qquad S\cdot\Delta p=S'\cdot\Delta p,' \quad i\cdot e.,\ S\cdot\Delta p=const.,$$

where Δp is the difference of pressure at the top and bottom of the layer. If the values of z, p, T and S in the new position be indicated by accented letters, the so-called Margules' formula can be written down as follows:

$$(6)\qquad \frac{1}{S'p'}\left(\frac{\partial T'}{\partial z'}+\Gamma\right)=\frac{1}{Sp}\left(\frac{\partial T}{\partial z}+\Gamma\right),\quad i\cdot e.,\ \frac{1}{Sp}\left(\frac{\partial T}{\partial z}+\Gamma\right)=const.$$

(See, Brunt, D., loc. cit. p. 45, Eq. 48)

Combining Eqs. (2), (4), (5), (6) and $p/\rho T=const.$ with Eq. (3), it is easily shown that

$$\frac{d}{dt}\left(\frac{f+\zeta}{\Delta p}\right)=0,$$

or

$$(f+\zeta)/\Delta p=const., \tag{7}$$

which is the same type with ROSSBY-HAURWITZ's equation for conservation of potential vorticity, and it is shown that *this conservation equation holds good even in the baroclinic atmosphere,* if it is permissible to introduce the isentropic and mass conservation approximations expressed by Eqs. (2) and (5)

An alternative proof by H. J. STEWART can be found in the "Handbook of Meteorology, edited by F. A. Berry, E.Bollay and Norman R. Beers, McGraw Hill, New York, 1945, pp.434-435."

The alternative expression of the ROSSBY-HAURWITZ's equation is

$$S\cdot(f+\zeta)=const., \tag{7·a}$$

under the same conditions.

Combining Eqs. (2), (4) and $p/\rho T=const.$ with Eq. (3), it is easily shown that

$$\frac{1}{p}\left(\frac{\partial T}{\partial z}+\Gamma\right)(f+\zeta)=const. \tag{3·a}$$

This is the alternative form of CHARNEY's equation of the conservation of vorticity, under the assumption that the motion is isentropic and frictionless. In the weather analysis, this expression (3·a) seems to be more convinient and more fruitful than the original expression.

References

〔1〕 CHARNEY, J.G., 1948; On the Scale of Atmospheric Motions, *Geof. Publ.*, **17**, No. 2, 17 pp.

〔2〕 CHARNEY, J.G., 1949; On a Physical Basis for Numerical Prediction of Large-Scale Motions in the Atmosphere, *J. Meteor.*, **6**. pp. 371-385.

〔3〕 HAURWITZ, B., 1941; *Dynamic Meteorology*, 1st. Ed., New York, Mc Graw-Hill Book Co. pp. 231-237.