The Conservation Law about the Horizontal of the Total Vorticity

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In view of the suggested importance of lateral mixing, C.=G. ROSSBY has given particular attention to the dynamics of the jet stream and its applications in meteorology and oceanography. But the conception of the absolute vorticity is often confused by several authors.(1) The present author intends to give particular attention to the horizontal component of the total vorticity, as well as the so-called "absolute vorticity" which normally means the component about the vertical of the total vorticity.

If we take a convinient set of axes of coordinates as follows:

x: horizontal and drawn to South,

y: horizontal and drawn to East,

z: vertical.

Let u, v, w, be the component velocities along these three axes, and ξ, η, ζ be the component vorticities along these three axes.

Then

$$\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\qquad \eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\qquad \zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}.$$

The well-known vorticity equations for barotropic fluid become

$$(1)\qquad \frac{d\xi}{dt}+(\xi-2\omega\cos\Phi)\,\Theta-\left\{(\xi-2\omega\cos\Phi)\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial y}+(\zeta+2\omega\sin\Phi)\frac{\partial u}{\partial z}\right\}=0,$$

$$(2)\qquad \frac{d\eta}{dt}+\eta\cdot\Theta\qquad-\left\{(\xi-2\omega\cos\Phi)\frac{\partial v}{\partial x}+\eta\frac{\partial v}{\partial y}+(\zeta+2\omega\sin\Phi)\frac{\partial v}{\partial z}\right\}=0,$$

$$(3)\qquad \frac{d\zeta}{dt}+(\zeta+2\omega\sin\Phi)\,\Theta-\left\{(\xi-2\omega\cos\Phi)\frac{\partial w}{\partial x}+\eta\frac{\partial w}{\partial y}+(\zeta+2\omega\sin\Phi)\frac{\partial w}{\partial z}\right\}=0,$$

where t is the time, ω the angular velocity of the earth's roation, Φ the latitude supposed to be constant and

$$\Theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z},$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

These expressions have been given by **Th. HESSELBERG** and **A. FRIEDMANN**(2). The equation of continuity may be written in the form

$$\frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0,$$

where ρ is the density. Upon substituting the expression for in Eqs. (1), (2) and (3), it follows that

$$(4)\qquad \rho\frac{d}{dt}\left(\frac{\xi-2\omega\cos\Phi}{\rho}\right)-\left\{(\xi-2\omega\cos\Phi)\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial y}+(\zeta+2\omega\sin\Phi)\frac{\partial u}{\partial z}\right\}=0,$$

$$(5)\qquad \rho\frac{d}{dt}\left(\frac{\eta}{\rho}\right)-\left\{(\xi-2\omega\cos\Phi)\frac{\partial v}{\partial x}+\eta\frac{\partial v}{\partial y}+(\zeta+2\omega\sin\Phi)\frac{\partial v}{\partial z}\right\}=0,$$

$$(6)\qquad \rho\frac{d}{dt}\left(\frac{\zeta+2\omega\sin\Phi}{\rho}\right)-\left\{(\zeta-2\omega\cos\Phi)\frac{\partial w}{\partial x}+\eta\frac{\partial w}{\partial y}+(\zeta+2\omega\sin\Phi)\frac{\partial w}{\partial z}\right\}=0.$$

For the convinient purpose, let the wind be oriented in the fixed direction at an angle β with the direction to South, at every level. Then we get the relation

$$(7)\qquad V=u\cos\beta+v\sin\beta,$$

$$(8)\qquad 0=-u\sin\beta+v\cos\beta,$$

where V is the wind speed.

Upon multiplying (5) with $\cos\beta$ and (4) with $\sin\beta$, and substraction, it is found that

$$\rho\frac{d}{dt}\left\{\frac{(\xi-2\omega\cos\Phi)(-\sin\beta)+\eta\cdot\cos\beta}{\rho}\right\}=0,$$

or

$$(9)\qquad \frac{(\xi-2\omega\cos\Phi)(-\sin\beta)+\eta\cdot\cos\beta}{\rho}=const.,$$

where $\eta\cdot\cos\beta-(\xi-2\omega\cos\Phi)\sin\beta$ is the horizontal component of the total absolute vorticity along the normal to the direction of wind. Although this vorticity component changes with the density of the layer, the expression (9) remains constant during the ascending motion or the descending motion. Thus, if the air is brought to a standard density ρ_0, the resulting vorticity $\eta_0\cdot\cos\beta-(\xi_0-2\omega\cos\Phi)\cdot\sin\beta$ may be considered as a characteristic, conservative property. The actual determination of the vorticity is of course difficult in view of the unreliability of the wind observations. Taking account of the order of magnitude, we can briefly put as follows:

$$\eta\cdot\cos\beta-(\xi-2\omega\cos\Phi)\cdot\sin\beta\approx\frac{\partial V}{\partial z},$$

so Eq. (9) is reduced to

$$(10)\qquad \frac{\partial V}{\partial z}\Big/\rho=\text{const.},\qquad \text{or}\qquad \left(\frac{\partial V}{\partial z}\right)\Big/\rho=\left(\frac{\partial V}{\partial z}\right)_0\Big/\rho_0.$$

Thus, if the air is brought to a standard density ρ_0 by ascending or descending motion, the resulting shear $\left(\frac{\partial V}{\partial z}\right)_0$ may be considered as a characteristic, conservative property, of which the terminology "*potential shear*" may be used.*

In extra-tropical region, the problem of wind-shear can not be understood without taking account of the thermal-wind law or the baroclinic condition. So the suggested conservative property, *potential shear*, will hold in the tropical region only.

* The above relation (9) or (10) may be obtained in another way. We take a set of the spherical polar coordinates as follows:

r : the radius,

ϕ : the geocentric latitude,

λ : the longitude.

Let V $(v_r,\ v_\phi,\ v_\lambda)$ be the component velocities along these three axes, where v_ϕ is conveniently assumed to be zero. The relative vorticity ω_ϕ along the $\phi-$ direction may be writte as

$$\omega_\phi=\frac{\partial v_\lambda}{\partial r}+\frac{v_\lambda}{r}-\frac{\partial v_r}{r\cos\phi\partial\lambda}.$$

The vorticity equation in spherical polar coordinates can be written in the form

$$\frac{d}{dt}(\omega_\phi+2\omega\cos\phi)+(\omega_\phi+2\omega\cos\phi)\,\Theta-\frac{v_r}{r}(\omega_\phi+2\omega\cos\phi)=0,$$

where

$$\Theta=\mathrm{div}\,V.$$

The equation of continuity may be written in the form

$$\frac{d\rho}{dt}+\rho\cdot\Theta=0.$$

Thus the equation of vorticity changes into

$$\rho\frac{d}{dt}\left(\frac{\omega_\phi+2\omega\cos\phi}{\rho}\right)-\frac{\frac{dr}{dt}}{r}(\omega_\phi+2\omega\cos\phi)=0,$$

or

$$\frac{1}{\left(\frac{\omega_\phi+2\omega\cos\phi}{\rho}\right)}\frac{d}{dt}\left(\frac{\omega_\phi+2\omega\cos\phi}{\rho}\right)-\frac{\frac{dr}{dt}}{r}=0.$$

If follows that

(9′) $$\phi\ \frac{2\omega\cos\phi}{\rho r}=\text{const.},$$

or nearly

(10′) $$\frac{\frac{\partial v_r}{\partial r}}{\rho}=\frac{\left(\frac{\partial v_r}{\partial r}\right)_0}{\rho_0},$$

which is the alternative proof of the relations (9) and (10) for the zonal circulation in the barotropic atmosphere.

References

(1) **Edward V. Ashburn:** Use of Geostrophic Wind and the Vorticity Equation in Cyclone Study, Journal of Meteorology, Vol. 2, No. 2, 132-133 (1945).

(2) **Th. Hesselberg und A. Friedmann;** Veröff. Geophys. Inst. Univ. Leipzig, 1 (1914), 147-173.