

The Vorticity Equations in Orthogonal Curvilinear Coordinates

by

H. Arakawa

Meteorological Research Institute

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Abstract

The transformation of the equations of motion of a viscous fluid to orthogonal curvilinear coordinates has been discussed by G. B. JEFFERY⁽¹⁾, and that of the equations of motion in dynamical meteorology to orthogonal curvilinear coordinates by the present author,⁽²⁾ but the analogous equations for the vorticity do not appear to have attracted the same attention. The first section of this paper deals with the transformation of the vorticity equations to the orthogonal curvilinear coordinates. The second section is devoted to applications to cylindrical and spherical polar coordinates.

1. Transformation of the Vorticity Equation

If V be the velocity, K the externally applied force, p the pressure, ρ the density, W the angular velocity of the earth's rotation, then the equation of motion in vectorial form is

$$\frac{dV}{dt} + 2[W \times V] = K - \frac{1}{\rho} \nabla p,$$

or

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V + 2[W \times V] = K - \frac{1}{\rho} \nabla p.$$

Taking the operation *CURL* on both sides of the above equation, and using the relations

$$\text{rot}\left(\frac{\partial V}{\partial t}\right) = \frac{\partial}{\partial t}(\text{rot } V),$$

$$\text{rot}\{(V \cdot \nabla)V\} = (V \cdot \nabla)\text{rot } V - (\text{rot } V \cdot \nabla)V + \text{rot } V \cdot \text{div } V,$$

$$\text{rot}[W \times V] = (V \cdot \nabla)W - (W \cdot \nabla)V + W \cdot \text{div } V - V \cdot \text{div } W,$$

$$\text{rot}\left(\frac{1}{\rho} \nabla p\right) = -\frac{1}{\rho^2}(\nabla \rho \times \nabla p),$$

$$\text{div } W = 0,$$

we get the vorticity equation as:⁽³⁾

$$(1) \quad \frac{\partial(\text{rot } V)}{\partial t} + (V \cdot \nabla)(\text{rot } V + 2W) + (\text{rot } V + 2W) \text{div } V$$

$$-\left\{(\operatorname{rot} V + 2W) \cdot \nabla\right\} V = \operatorname{rot} K + \frac{1}{\rho^2} (\nabla \rho \times \nabla p).$$

Again using the relations

$$\operatorname{rot}\left\{(\operatorname{rot} V + 2W) \times V\right\} = (V \cdot \nabla)(\operatorname{rot} V + 2W) - \left\{(\operatorname{rot} V + 2W) \cdot \nabla\right\} V \\ + (\operatorname{rot} V + 2W) \cdot \operatorname{div} V - V \cdot \operatorname{div}(\operatorname{rot} V + 2W),$$

and $\frac{\partial W}{\partial t} = 0, \quad \operatorname{div} W = 0, \quad \operatorname{div} \cdot \operatorname{rot} V = 0,$

we get the vorticity equation as follows:

$$(1.a) \quad \frac{\partial(\operatorname{rot} V + 2W)}{\partial t} + \operatorname{rot}\left\{(\operatorname{rot} V + 2W) \times V\right\} = \operatorname{rot} K + \frac{1}{\rho^2} (\nabla \rho \times \nabla p).$$

The transformations of *div* and *rot* follow at once from the definition of these operators in terms of surface and line integrals, respectively.

If α, β, γ are orthogonal curvilinear coordinates, and if

$$h_1^2 = \left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2 + \left(\frac{\partial \alpha}{\partial z}\right)^2, \quad h_2^2 = \left(\frac{\partial \beta}{\partial x}\right)^2 + \left(\frac{\partial \beta}{\partial y}\right)^2 + \left(\frac{\partial \beta}{\partial z}\right)^2, \\ h_3^2 = \left(\frac{\partial \gamma}{\partial x}\right)^2 + \left(\frac{\partial \gamma}{\partial y}\right)^2 + \left(\frac{\partial \gamma}{\partial z}\right)^2,$$

so that elements of arc measured along the normals to the coordinate surfaces at any point are $\frac{\delta \alpha}{h_1}, \frac{\delta \beta}{h_2}, \frac{\delta \gamma}{h_3}$, then

$$(2) \quad \operatorname{rot} V \left[\zeta_1 = h_2 h_3 \left\{ \frac{\partial}{\partial \beta} \left(\frac{v_3}{h_3} \right) - \frac{\partial}{\partial \gamma} \left(\frac{v_2}{h_2} \right) \right\}, \quad \zeta_2 = h_3 h_1 \left\{ \frac{\partial}{\partial \gamma} \left(\frac{v_1}{h_1} \right) - \frac{\partial}{\partial \alpha} \left(\frac{v_3}{h_3} \right) \right\}, \right. \\ \left. \zeta_3 = h_1 h_2 \left\{ \frac{\partial}{\partial \alpha} \left(\frac{v_2}{h_2} \right) - \frac{\partial}{\partial \beta} \left(\frac{v_1}{h_1} \right) \right\} \right],$$

$$(3) \quad \operatorname{div} V = h_1 h_2 h_3 \left\{ \frac{\partial}{\partial \alpha} \left(\frac{v_1}{h_2 h_3} \right) + \frac{\partial}{\partial \beta} \left(\frac{v_2}{h_3 h_1} \right) + \frac{\partial}{\partial \gamma} \left(\frac{v_3}{h_1 h_2} \right) \right\},$$

where v_1, v_2, v_3 are the components of V along the normals to the surfaces $\alpha, \beta, \gamma = \text{const.}$, respectively, and $\zeta_1, \zeta_2, \zeta_3$ the components of $\operatorname{rot} V$ along the normals $\alpha, \beta, \gamma = \text{const.}$, respectively.

By a second application of the operations implied in (2), we have, denoting the direction of a component by a suffix,

$$\operatorname{rot}_\alpha \left\{ (\operatorname{rot} V + 2W) \times V \right\} = h_2 h_3 \left[\frac{\partial}{\partial \beta} \left\{ \frac{1}{h_3} [(\zeta_1 + 2\omega_1)v_2 - (\zeta_2 + 2\omega_2)v_1] \right\} \right. \\ \left. - \frac{\partial}{\partial \gamma} \left\{ \frac{1}{h_2} [(\zeta_3 + 2\omega_3)v_1 - (\zeta_1 + 2\omega_1)v_3] \right\} \right].$$

Using the relations

$$\operatorname{div} W = h_1 h_2 h_3 \left\{ \frac{\partial}{\partial \alpha} \left(\frac{\omega_1}{h_2 h_3} \right) + \frac{\partial}{\partial \beta} \left(\frac{\omega_2}{h_3 h_1} \right) + \frac{\partial}{\partial \gamma} \left(\frac{\omega_3}{h_1 h_2} \right) \right\} \equiv 0,$$

$$\operatorname{div} \cdot \operatorname{rot} V = h_1 h_2 h_3 \left\{ \frac{\partial}{\partial \alpha} \left(\frac{\zeta_1}{h_2 h_3} \right) + \frac{\partial}{\partial \beta} \left(\frac{\zeta_2}{h_3 h_1} \right) + \frac{\partial}{\partial \gamma} \left(\frac{\zeta_3}{h_1 h_2} \right) \right\} \equiv 0,$$

the last relation can be transformed into an expression which while it appears more complex, is in reality simpler in application, namely,

$$(4) \quad \left\{ \begin{aligned} \text{rot} \{ (\text{rot} V + 2W) \times V \} &= (v_1 h_1 \frac{\partial}{\partial \alpha} + v_2 h_2 \frac{\partial}{\partial \beta} + v_3 h_3 \frac{\partial}{\partial \gamma}) (\zeta_1 + 2\omega_1) \\ &+ (\zeta_1 + 2\omega_1) h_1 h_2 h_3 \left\{ \frac{\partial}{\partial \alpha} \left(\frac{v_1}{h_2 h_3} \right) + \frac{\partial}{\partial \beta} \left(\frac{v_2}{h_3 h_1} \right) + \frac{\partial}{\partial \gamma} \left(\frac{v_3}{h_1 h_2} \right) \right\} \\ &- \left\{ (\zeta_1 + 2\omega_1) h_1 \frac{\partial}{\partial \alpha} + (\zeta_2 + 2\omega_2) h_2 \frac{\partial}{\partial \beta} + (\zeta_3 + 2\omega_3) h_3 \frac{\partial}{\partial \gamma} \right\} v_1 \\ &- \left\{ (\zeta_1 + 2\omega_1) v_2 - (\zeta_2 + 2\omega_2) v_1 \right\} h_1 h_2 \frac{\partial}{\partial \beta} \left(\frac{1}{h_1} \right) + \left\{ (\zeta_3 + 2\omega_3) v_1 \right. \\ &\left. - (\zeta_1 + 2\omega_1) v_3 \right\} h_1 h_3 \frac{\partial}{\partial \gamma} \left(\frac{1}{h_1} \right). \end{aligned} \right.$$

Let

$$(5) \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_1 h_1 \frac{\partial}{\partial \alpha} + v_2 h_2 \frac{\partial}{\partial \beta} + v_3 h_3 \frac{\partial}{\partial \gamma},$$

then the α component of (1.a) becomes

$$(6) \quad \left\{ \begin{aligned} &\frac{d(\zeta_1 + 2\omega_1)}{dt} + (\zeta_1 + 2\omega_1) h_1 h_2 h_3 \left\{ \frac{\partial}{\partial \alpha} \left(\frac{v_1}{h_2 h_3} \right) + \frac{\partial}{\partial \beta} \left(\frac{v_2}{h_3 h_1} \right) + \frac{\partial}{\partial \gamma} \left(\frac{v_3}{h_1 h_2} \right) \right\} \\ &- \left\{ (\zeta_1 + 2\omega_1) h_1 \frac{\partial}{\partial \alpha} + (\zeta_2 + 2\omega_2) h_2 \frac{\partial}{\partial \beta} + (\zeta_3 + 2\omega_3) h_3 \frac{\partial}{\partial \gamma} \right\} v_1 \\ &- \left\{ (\zeta_1 + 2\omega_1) v_2 - (\zeta_2 + 2\omega_2) v_1 \right\} h_1 h_2 \frac{\partial}{\partial \beta} \left(\frac{1}{h_1} \right) \\ &\quad + \left\{ (\zeta_3 + 2\omega_3) v_1 - (\zeta_1 + 2\omega_1) v_3 \right\} h_1 h_3 \frac{\partial}{\partial \gamma} \left(\frac{1}{h_1} \right) \\ &= h_2 h_3 \left\{ \frac{\partial}{\partial \beta} \left(\frac{K_3}{h_3} \right) - \frac{\partial}{\partial \gamma} \left(\frac{K_2}{h_2} \right) \right\} + \frac{1}{\rho^2} h_2 h_3 \left(\frac{\partial \rho}{\partial \beta} \frac{\partial p}{\partial \gamma} - \frac{\partial \rho}{\partial \gamma} \frac{\partial p}{\partial \beta} \right). \end{aligned} \right.$$

If the externally applied force K (K_1, K_2, K_3) has a potential φ , then using the relation $\text{rot} K = \text{rot} \cdot \text{grad} \varphi = 0$, the final equation (6) becomes more simple. The corresponding equations in ζ_2, ζ_3 can at once be written down from symmetry.

2. Application to Cylindrical and Spherical Polar Coordinates

If we take Cartesian coordinates x, y, z , with the x -axis drawn to *south*, the y -axis drawn to *east* and the z -axis vertically upwards, the component vorticities along these three axes are given by

$$\zeta_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \quad \zeta_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \quad \zeta_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y},$$

where v_x, v_y, v_z are the component velocities. The vorticity equations in Cartesian coordinates become

$$\begin{aligned} &\frac{d}{dt} (\zeta_x - 2\omega \cos \theta) + (\zeta_x - 2\omega \cos \theta) \cdot \theta - \left\{ (\zeta_x - 2\omega \cos \theta) \frac{\partial v_x}{\partial x} \right. \\ &\quad \left. + \zeta_y \frac{\partial v_x}{\partial y} + (\zeta_z + 2\omega \sin \theta) \frac{\partial v_x}{\partial z} \right\} = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial y} \frac{\partial p}{\partial z} - \frac{\partial \rho}{\partial z} \frac{\partial p}{\partial y} \right), \\ &\frac{d}{dt} \zeta_y + \zeta_y \cdot \theta - \left\{ (\zeta_x - 2\omega \cos \theta) \frac{\partial v_y}{\partial x} \right. \\ &\quad \left. + \zeta_y \frac{\partial v_y}{\partial y} + (\zeta_z + 2\omega \sin \theta) \frac{\partial v_y}{\partial z} \right\} = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial z} \frac{\partial p}{\partial x} - \frac{\partial \rho}{\partial x} \frac{\partial p}{\partial z} \right), \\ &\frac{d}{dt} (\zeta_z + 2\omega \sin \theta) + (\zeta_z + 2\omega \sin \theta) \cdot \theta - \left\{ (\zeta_x - 2\omega \cos \theta) \frac{\partial v_z}{\partial x} \right. \\ &\quad \left. + \zeta_y \frac{\partial v_z}{\partial y} + (\zeta_z + 2\omega \sin \theta) \frac{\partial v_z}{\partial z} \right\} = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y} \frac{\partial p}{\partial x} \right). \end{aligned}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_s \frac{\partial}{\partial x} + v_\theta \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z},$$

θ is the latitude supposed to be a constant and $\theta = \text{div } V$. These expressions have been given by **Th. HESSELBERG** and **A. FRIEDMANN**.⁽⁴⁾

If we take cylindrical coordinates s, θ, z , we have

$$x = s \cos \theta, \quad y = s \sin \theta, \quad z = z, \quad h_1 = h_3 = 1 \quad \text{and} \quad h_2 = \frac{1}{s}.$$

Then the component vorticities along s -, θ -, z -axis are given by

$$\zeta_s = \frac{1}{s} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \quad \zeta_\theta = \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}, \quad \zeta_z = \frac{\partial v_\theta}{\partial s} + \frac{v_\theta}{s} - \frac{\partial v_s}{s \partial \theta},$$

where v_s, v_θ, v_z are the component velocities. From (6) the vorticity equations in cylindrical coordinates become

$$\begin{aligned} & \frac{d}{dt} (\zeta_s - 2\omega \cos \theta \cos \theta) + (\zeta_s - 2\omega \cos \theta \cos \theta) \cdot \theta \\ & - \left[(\zeta_s - 2\omega \cos \theta \cos \theta) \frac{\partial v_s}{\partial s} + (\zeta_\theta + 2\omega \cos \theta \sin \theta) \frac{\partial v_s}{s \partial \theta} + (\zeta_z + 2\omega \sin \theta) \frac{\partial v_s}{\partial z} \right] \\ & = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial \theta} \frac{\partial p}{\partial z} - \frac{\partial \rho}{\partial z} \frac{\partial p}{\partial \theta} \right), \\ & \frac{d}{dt} (\zeta_\theta + 2\omega \cos \theta \sin \theta) + (\zeta_\theta + 2\omega \cos \theta \sin \theta) \cdot \theta \\ & - \left[(\zeta_s - 2\omega \cos \theta \cos \theta) \frac{\partial v_\theta}{\partial s} + (\zeta_\theta + 2\omega \cos \theta \sin \theta) \frac{\partial v_\theta}{s \partial \theta} + (\zeta_z + 2\omega \sin \theta) \frac{\partial v_\theta}{\partial z} \right] \\ & + \frac{v_\theta}{s} (\zeta_s - 2\omega \cos \theta \cos \theta) - \frac{v_s}{s} (\zeta_\theta + 2\omega \cos \theta \sin \theta) = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial z} \frac{\partial p}{\partial s} - \frac{\partial \rho}{\partial s} \frac{\partial p}{\partial z} \right), \\ & \frac{d}{dt} (\zeta_z + 2\omega \sin \theta) + (\zeta_z + 2\omega \sin \theta) \cdot \theta \\ & - \left[(\zeta_s - 2\omega \cos \theta \cos \theta) \frac{\partial v_z}{\partial s} + (\zeta_\theta + 2\omega \cos \theta \sin \theta) \frac{\partial v_z}{s \partial \theta} + (\zeta_z + 2\omega \sin \theta) \frac{\partial v_z}{\partial z} \right] \\ & = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial s} \frac{\partial p}{\partial \theta} - \frac{\partial \rho}{\partial \theta} \frac{\partial p}{\partial s} \right), \end{aligned}$$

where

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial t} + v_s \frac{\partial}{\partial s} + v_\theta \frac{\partial}{s \partial \theta} + v_z \frac{\partial}{\partial z}, \\ \theta &= \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}, \end{aligned}$$

and $\zeta_s - 2\omega \cos \theta \cos \theta$, $\zeta_\theta + 2\omega \cos \theta \sin \theta$, $\zeta_z + 2\omega \sin \theta$, are the components of absolute vorticity along s -, θ -, z -axis, respectively. These expressions have been discussed by the present author.⁽⁵⁾

If r, θ, λ are the spherical polar coordinates with the geocentric colatitude θ and longitude λ , then $h_1 = 1$, $h_2 = \frac{1}{r}$, $h_3 = \frac{1}{r \sin \theta}$. The component vorticities along r -, θ -, λ -axis are given by

$$\zeta_r = \frac{1}{r \sin \theta} \left\{ \frac{\partial}{\partial \theta} (\sin \theta \cdot v_\lambda) - \frac{\partial v_\theta}{\partial \lambda} \right\},$$

$$\zeta_\theta = \frac{1}{r} \left\{ \frac{1}{\sin\theta} \frac{\partial v_r}{\partial \lambda} - \frac{\partial}{\partial r} (rv_\lambda) \right\};$$

$$\zeta_\lambda = \frac{1}{r} \left\{ \frac{\partial (rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right\},$$

where v_r, v_θ, v_λ are the component velocities. From (6), the vorticity equations in spherical polar coordinates become

$$\begin{aligned} & \frac{d}{dt} (\zeta_r + 2\omega \cos\theta) + (\zeta_r + 2\omega \cos\theta) \cdot \theta \\ & - \left\{ (\zeta_r + 2\omega \cos\theta) \frac{\partial v_r}{\partial r} + (\zeta_\theta - 2\omega \sin\theta) \frac{\partial v_r}{r \partial \theta} + \zeta_\lambda \frac{\partial v_r}{r \sin\theta \partial \lambda} \right\} \\ & = \frac{1}{\rho^2} \left(\frac{\partial \rho}{r \partial \theta} \frac{\partial p}{r \sin\theta \partial \lambda} - \frac{\partial \rho}{r \sin\theta \partial \lambda} \frac{\partial p}{r \partial \theta} \right), \\ & \frac{d}{dt} (\zeta_\theta - 2\omega \sin\theta) + (\zeta_\theta - 2\omega \sin\theta) \cdot \theta \\ & - \left\{ (\zeta_r + 2\omega \cos\theta) \frac{\partial v_\theta}{\partial r} + (\zeta_\theta - 2\omega \sin\theta) \frac{\partial v_\theta}{r \partial \theta} + \zeta_\lambda \frac{\partial v_\theta}{r \sin\theta \partial \lambda} \right\} \\ & - \frac{v_r}{r} (\zeta_\theta - 2\omega \sin\theta) + \frac{v_\theta}{r} (\zeta_r + 2\omega \cos\theta) \\ & = \frac{1}{\rho^2} \left(\frac{\partial \rho}{r \sin\theta \partial \lambda} \frac{\partial p}{\partial r} - \frac{\partial \rho}{\partial r} \frac{\partial p}{r \sin\theta \partial \lambda} \right), \\ & \frac{d\zeta_\lambda}{dt} + \zeta_\lambda \theta - \left\{ (\zeta_r + 2\omega \cos\theta) \frac{\partial v_\lambda}{\partial r} + (\zeta_\theta - 2\omega \sin\theta) \frac{\partial v_\lambda}{r \partial \theta} + \zeta_\lambda \frac{\partial v_\lambda}{r \sin\theta \partial \lambda} \right\} \\ & - \frac{v_r}{r} \zeta_\lambda + \frac{v_\lambda}{r} (\zeta_r + 2\omega \cos\theta) + \frac{\cot\theta}{r} v_\lambda (\zeta_\theta - 2\omega \sin\theta) - \frac{\cot\theta}{r} v_\theta \cdot \zeta_\lambda \\ & = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial r} \frac{\partial p}{r \partial \theta} - \frac{\partial \rho}{r \partial \theta} \frac{\partial p}{\partial r} \right), \end{aligned}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + \frac{v_\lambda}{r \sin\theta} \frac{\partial}{\partial \lambda},$$

$$\theta = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (v_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial v_\lambda}{\partial \lambda},$$

and $\zeta_r + 2\omega \cos\theta, \zeta_\theta - 2\omega \sin\theta, \zeta_\lambda$ are the components of absolute vorticity along r, θ, λ -axis, respectively. These expressions have been discussed by the present author.⁽⁵⁾

Reference

- 1). G. B. JEFFERY: The Equations of Motion of a Viscous Fluid, Phil. Mag., Ser. 6, Vol. 29, pp. 445-455 (1915).
- 2). H. ARAKAWA: Transformation of the Equations of Motion in Dynamical Meteorology to Orthogonal Curvilinear Coordinates, Papers in Meteorology and Geophysics, Vol. 1, 45-49 (1950).
- 3). H. ARAKAWA: Die Wirbelgleichungen mit Berücksichtigung der Erddrehung, Meteorol. Zeitschr., 1941, S. 70-71.
- 4). Th. HESSELBERG und A. FRIEDMANN: Die Größenordnung der meteorologischen Elemente und ihrer räumlichen und zeitlichen Ableitungen, Veröff. Geophys. Inst. Univ. Leipzig, Bd. 1, S. 147-173, 1914.
- 5). H. ARAKAWA: The Vorticity Equations in the Spherical and Cylindrical Coordinates, Geophys. Mag., Tokyo, Vol. 16, pp. 1-4 (1948).