Transformation of the Equations of Motion in Dynamical Meteorology to Orthogonal Curvilinear Coordinates

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Abstract

The transformation of the equations of motion of a viscous fluid to orthogonal curvilinear coordinates has been discussed by G. B. JEFFERY, [1] and others [2][3], but analogous equations for the motion in dynamical meteorology do not appear to have attracted the same attention. The first section of this paper deals with the transformation of the equations of motion in dynamical meteorology. Then the theory is illustrated by applications to cylindrical and spherical polar coordinates.

1. Transformation of the Equation of Motion

If $\mathbf{V}$ be the velocity, $\mathbf{K}$ the externally applied body force respectively, t the time, p the mean pressure, ρ the density, and μ the coefficient of viscosity, $\mathbf{W}$ the angular velocity of the earth's rotation, then the equations of motion in dynamical meteorology are

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}+2(\mathbf{W}\times\mathbf{V})=\mathbf{K}-\frac{1}{\rho}\,\mathrm{grad}\;p+\frac{1}{3}\frac{\mu}{\rho}\,\mathrm{grad}\cdot\mathrm{div}\;\mathbf{V}+\frac{\mu}{\rho}\nabla^2\mathbf{V}.$$

Using the relations

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}=\frac{\partial\mathbf{V}}{\partial t}+\frac{1}{2}\,\mathrm{grad}\,(\mathbf{V}^2)-(\mathbf{V}\times\mathrm{rot}\;\mathbf{V}),$$

$$\mathrm{rot}\cdot\mathrm{rot}\;\mathbf{V}=\mathrm{grad}\cdot\mathrm{div}\;\mathbf{V}-\nabla^2\mathbf{V},$$

the vectorial equation of motion is

$$(1)\qquad \frac{\partial\mathbf{V}}{\partial t}+\frac{1}{2}\,\mathrm{grad}\,(\mathbf{V}^2)-(\mathbf{V}\times\mathrm{rot}\;\mathbf{V})+2(\mathbf{W}\times\mathbf{V})$$

$$=\mathbf{K}-\frac{1}{\rho}\,\mathrm{grad}\;p+\frac{4}{3}\frac{\mu}{\rho}\,\mathrm{grad}\cdot\mathrm{div}\;\mathbf{V}-\frac{\mu}{\rho}\,\mathrm{rot}\cdot\mathrm{rot}\;\mathbf{V}.$$

The transformations of *div* and *rot* follow at once from the definition of these operators in terms of surface and line integrals, respectively.

If α, β, γ are orthogonal curvilinear coordinates, and if

$$h_1^2=\left(\frac{\partial\alpha}{\partial x}\right)^2+\left(\frac{\partial\alpha}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial z}\right)^2,\quad h_2^2=\left(\frac{\partial\beta}{\partial x}\right)^2+\left(\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\beta}{\partial z}\right)^2,$$

$$h_3^2=\left(\frac{\partial\gamma}{\partial x}\right)^2+\left(\frac{\partial\gamma}{\partial y}\right)^2+\left(\frac{\partial\gamma}{\partial z}\right)^2,$$

so that elements of are measured along normals to the coordinate surfaces at any point are

$$\frac{\partial \alpha}{h_1},\ \frac{\partial \beta}{h_2},\ \frac{\partial \gamma}{h_3},$$

then

$$(2)\qquad \mathrm{rot}\,\mathbf{V}=h_2h_3\left\{\frac{\partial}{\partial\beta}\left(\frac{v_\gamma}{h_3}\right)-\frac{\partial}{\partial\gamma}\left(\frac{v_\beta}{h_2}\right)\right\},\ h_3h_1\left\{\frac{\partial}{\partial\gamma}\left(\frac{v_\alpha}{h_1}\right)-\frac{\partial}{\partial\alpha}\left(\frac{v_\gamma}{h_3}\right)\right\},$$

$$h_1h_2\left\{\frac{\partial}{\partial\alpha}\left(\frac{v_\beta}{h_2}\right)-\frac{\partial}{\partial\beta}\left(\frac{v_\alpha}{h_1}\right)\right\},$$

$$(3)\qquad \mathrm{div}\,\mathbf{V}=h_1h_2h_3\left\{\frac{\partial}{\partial\alpha}\left(\frac{v_\alpha}{h_2h_3}\right)+\frac{\partial}{\partial\beta}\left(\frac{v_\beta}{h_3h_1}\right)+\frac{\partial}{\partial\gamma}\left(\frac{v_\gamma}{h_1h_2}\right)\right\},$$

where v_α, v_β, v_γ are the components of $\mathbf{V}$ along the normals to the surfaces $\alpha,\beta,\gamma=const.$, respectively. In (3) put $\mathbf{V}=\mathrm{grad}\,\phi$, and we obtain

$$(4)\qquad \nabla^2\phi_1=h_1h_2h_3\left\{\frac{\partial}{\partial\alpha}\left(\frac{h_1}{h_2h_3}\frac{\partial\phi}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{h_2}{h_3h_1}\frac{\partial\phi}{\partial\beta}\right)+\frac{\partial}{\partial\gamma}\left(\frac{h_3}{h_1h_2}\frac{\partial\phi}{\partial\gamma}\right)\right\}.$$

By a second application of the operations implied in (2) we have, denoting the direction of a compoment by a suffix,

$$\mathrm{rot}_\alpha\cdot\mathrm{rot}\,\mathbf{V}=h_2h_3\left[\frac{\partial}{\partial\beta}\left\{\frac{h_1h_2}{h_3}\left(\frac{\partial}{\partial\alpha}\left(\frac{v_\beta}{h_2}\right)-\frac{\partial}{\partial\beta}\left(\frac{v_\alpha}{h_1}\right)\right)\right\}\right.$$

$$\left.-\frac{\partial}{\partial\gamma}\left\{\frac{h_3h_1}{h_2}\left(\frac{\partial}{\partial\gamma}\left(\frac{v_\alpha}{h_1}\right)-\frac{\partial}{\partial\alpha}\left(\frac{v_\gamma}{h_3}\right)\right)\right\}\right].$$

This can be transformed into an expression which is originally given by G. B. JEFFERY, namely

$$(5)\qquad \mathrm{rot}_\alpha\,\mathrm{rot}\,\mathbf{V}=h_1\frac{\partial}{\partial\alpha}(\mathrm{div}\,\mathbf{V})-\nabla^2v_\alpha+\frac{v_\alpha}{h_1}\nabla^2h_1-v_\alpha\frac{\partial}{\partial\alpha}(\nabla^2\alpha)-\frac{h_1v_\beta}{h_2}\frac{\partial}{\partial\alpha}(\nabla^2\beta)$$

$$-\frac{h_1v_\gamma}{h_3}\frac{\partial}{\partial\alpha}(\nabla^2\gamma)+2h_2^2\frac{\partial h_1}{\partial\beta}\frac{\partial}{\partial\alpha}\left(\frac{v_\beta}{h_2}\right)-2h_1h_2\frac{\partial h_2}{\partial\alpha}\frac{\partial}{\partial\beta}\left(\frac{v_\beta}{h_2}\right)$$

$$+2h_3^2\frac{\partial h_1}{\partial\gamma}\frac{\partial}{\partial\alpha}\left(\frac{v_\gamma}{h_3}\right)-2h_1h_3\frac{\partial h_3}{\partial\alpha}\frac{\partial}{\partial\gamma}\left(\frac{v_\gamma}{h_3}\right).$$

It remains to find appropriate expressions for the components of $\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t}+2(\mathbf{W}\times\mathbf{V})$. Taking the component along the normal to the surface $\alpha=const.$, we get

$$\left(\frac{\partial\mathbf{V}}{\partial t}\right)_\alpha=\frac{\partial v_\alpha}{\partial t},$$

$$\left(\frac{1}{2}\,\mathrm{grad}\,(\mathbf{V}^2)\right)_\alpha=h_1\left(v_\alpha\frac{\partial v_\alpha}{\partial\alpha}+v_\beta\frac{\partial v_\beta}{\partial\alpha}+v_\gamma\frac{\partial v_\gamma}{\partial\alpha}\right),$$

$$(\mathbf{V}\times\mathrm{rot}\,\mathbf{V})_\alpha=v_\beta\cdot h_1h_2\left\{\frac{\partial}{\partial\alpha}\left(\frac{v_\beta}{h_2}\right)-\frac{\partial}{\partial\beta}\left(\frac{v_\alpha}{h_1}\right)\right\}-v_\gamma\cdot h_3h_1\left\{\frac{\partial}{\partial\gamma}\left(\frac{v_\alpha}{h_1}\right)-\frac{\partial}{\partial\alpha}\left(\frac{v_\gamma}{h_3}\right)\right\},$$

$$2(\mathbf{W}\times\mathbf{V})_\alpha=2\,\omega_\beta v_\gamma-2\,\omega_\gamma v_\beta,$$

where ω_α, ω_β, ω_γ are the components of **W** along the normals to the surfaces α, β, $\gamma=const.$, respectively. Combining these, the component acceleration $\frac{D\mathbf{V}}{Dt}+2[\mathbf{W}\times\mathbf{V}]$ along the normal to the surface $\alpha=const.$, is

$$\frac{\partial v_\alpha}{\partial t}+h_1 v_\alpha\frac{\partial v_\alpha}{\partial \alpha}+h_2 v_\beta\frac{\partial v_\alpha}{\partial \beta}+h_3 v_\gamma\frac{\partial v_\alpha}{\partial \gamma}$$

$$+h_1 v_\alpha\left\{h_1 v_\alpha\frac{\partial}{\partial\alpha}\left(\frac{1}{h_1}\right)+h_2 v_\beta\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)+h_3 v_\gamma\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)\right\}$$

$$-h_1\left\{h_1 v_\alpha^2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_1}\right)+h_2 v_\beta^2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)+h_3 v_\gamma^2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)\right\}$$

$$+2\,\omega_\beta v_\gamma-2\,\omega_\gamma v_\beta.$$

The remaining components can be written down by symmetry.

Thus the α component of Eq. (1) becomes

$$(6a)\quad \frac{dv_\alpha}{dt}+h_1 v_\alpha\left\{h_2 v_\beta\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)+h_3 v_\gamma\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)\right\}-h_1h_2v_\beta^2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)$$

$$-h_1h_3v_\gamma^2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)+2\,\omega_\beta v_\gamma-2\,\omega_\gamma v_\beta$$

$$=K_\alpha-\frac{h_1}{\rho}\frac{\partial p}{\partial\alpha}+\frac{\mu}{\rho}\left[\frac{h_1}{3}\frac{\partial\Theta}{\partial\alpha}+\nabla^2 v_\alpha-\frac{v_\alpha}{h_1}\nabla^2 h_1\right.$$

$$+v_\alpha\frac{\partial}{\partial\alpha}(\nabla^2\alpha)+\frac{h_1 v_\beta}{h_2}\frac{\partial}{\partial\alpha}(\nabla^2\beta)+\frac{h_1 v_\gamma}{h_3}\frac{\partial}{\partial\alpha}(\nabla^2\gamma)$$

$$-2h_2^2\frac{\partial h_1}{\partial\beta}\frac{\partial}{\partial\alpha}\left(\frac{v_\beta}{h_2}\right)+2h_1h_2\frac{\partial h_2}{\partial\alpha}\frac{\partial}{\partial\beta}\left(\frac{v_\beta}{h_2}\right)-2h_3^2\frac{\partial h_1}{\partial\gamma}\frac{\partial}{\partial\alpha}\left(\frac{v_\gamma}{h_3}\right)$$

$$\left.+2h_1h_3\frac{\partial h_3}{\partial\alpha}\frac{\partial}{\partial\gamma}\left(\frac{v_\gamma}{h_3}\right)\right]$$

where

$$\Theta=\operatorname{div}\mathbf{V},\ \text{and}\quad \frac{d}{dt}\equiv\frac{\partial}{\partial t}+h_1 v_\alpha\frac{\partial}{\partial\alpha}+h_2 v_\beta\frac{\partial}{\partial\beta}+h_3 v_\gamma\frac{\partial}{\partial\gamma}.$$

The corresponding equations in v_β, v_γ can at once be written down from symmetry as;

$$(6b)\quad \frac{dv_\beta}{dt}+h_2 v_\beta\left\{h_3 v_\gamma\frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\right)+h_1 v_\alpha\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\right\}-h_2h_3v_\gamma^2\frac{\partial}{\partial\beta}\left(\frac{1}{h_3}\right)$$

$$-h_2h_1v_\alpha^2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)+2\,\omega_\gamma v_\alpha-2\,\omega_\alpha v_\gamma$$

$$=K_\beta-\frac{h_2}{\rho}\frac{\partial p}{\partial\beta}+\frac{\mu}{\rho}\left[\frac{h_2}{3}\frac{\partial\Theta}{\partial\beta}+\nabla^2 v_\beta-\frac{v_\beta}{h_2}\nabla^2 h_2\right.$$

$$+v_\beta\frac{\partial}{\partial\beta}(\nabla^2\beta)+\frac{h_2 v_\gamma}{h_3}\frac{\partial}{\partial\beta}(\nabla^2\gamma)+\frac{h_2 v_\alpha}{h_1}\frac{\partial}{\partial\beta}(\nabla^2\alpha)$$

$$-2h_3^2\frac{\partial h_2}{\partial \gamma}\frac{\partial}{\partial \beta}\left(\frac{v_\gamma}{h_3}\right)+2h_2h_3\frac{\partial h_3}{\partial \beta}\frac{\partial}{\partial \gamma}\left(\frac{v_\gamma}{h_3}\right)-2h_1^2\frac{\partial h_2}{\partial \alpha}\frac{\partial}{\partial \beta}\left(\frac{v_\alpha}{h_1}\right)$$

$$+2h_2h_1\frac{\partial h_1}{\partial \beta}\frac{\partial}{\partial \alpha}\left(\frac{v_\alpha}{h_1}\right)\Big],$$

and

$$(6c)\quad \frac{dv_\gamma}{dt}+h_3v_\gamma\left\{h_1v_\alpha\frac{\partial}{\partial \alpha}\left(\frac{1}{h_3}\right)+h_2v_\beta\frac{\partial}{\partial \beta}\left(\frac{1}{h_3}\right)\right\}-h_3h_1v_\alpha^2\frac{\partial}{\partial \gamma}\left(\frac{1}{h_1}\right)$$

$$-h_3h_2v_\beta^2\frac{\partial}{\partial \gamma}\left(\frac{1}{h_2}\right)+2\,\omega_\alpha v_\beta-2\,\omega_\beta v_\alpha$$

$$=K_\gamma-\frac{h_3}{\rho}\frac{\partial p}{\partial \gamma}+\frac{\mu}{\rho}\Big[\frac{h_3}{3}\frac{\partial \Theta}{\partial \gamma}+\nabla^2v_\gamma-\frac{v_\gamma}{h_3}\nabla^2h_3+v_\gamma\frac{\partial}{\partial \gamma}(\nabla^2\gamma)$$

$$+\frac{h_3v_\alpha}{h_1}\frac{\partial}{\partial \gamma}(\nabla^2\alpha)+\frac{h_3v_\beta}{h_2}\frac{\partial}{\partial \gamma}(\nabla^2\beta)-2h_1^2\frac{\partial h_3}{\partial \alpha}\frac{\partial}{\partial \gamma}\left(\frac{v_\alpha}{h_1}\right)$$

$$+2h_3h_1\frac{\partial h_1}{\partial \gamma}\frac{\partial}{\partial \alpha}\left(\frac{v_\alpha}{h_1}\right)-2h_2^2\frac{\partial h_3}{\partial \beta}\frac{\partial}{\partial \gamma}\left(\frac{v_\beta}{h_2}\right)+2h_3h_2\frac{\partial h_2}{\partial \gamma}\frac{\partial}{\partial \beta}\left(\frac{v_\beta}{h_2}\right)\Big].$$

2. Application to Cylindrical and Spherical Polar Coordinates

If we take cylindrical coordinates s, θ, z and the axis of z is taken vertically upwards, we have

$$h_1=1,\ h_2=1/s,\ \text{and}\ h_3=1.$$

If the horizontal initial axis $\theta=0$ be drawn to South, we then have the following expressions

$$\omega_s=-\omega\cos\Phi\cdot\cos\theta,\quad \omega_\theta=\omega\cos\Phi\sin\theta,\quad \omega_z=\omega\sin\Phi,$$

where ω is the angular velocity of the earth's rotation and Φ the latitude. Most of the terms in Eqs. (6) vanish and we have

$$\frac{dv_s}{dt}-\frac{v_\theta^2}{s}+2\,\omega\cos\Phi\sin\theta\cdot v_z-2\,\omega\sin\Phi\cdot v_\theta$$

$$=K_s-\frac{1}{\rho}\frac{\partial p}{\partial s}+\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial \Theta}{\partial s}+\nabla^2v_s-\frac{v_s}{s^2}-\frac{2}{s^2}\frac{\partial v_\theta}{\partial \theta}\right],$$

$$\frac{dv_\theta}{dt}+\frac{v_sv_\theta}{s}+2\,\omega\sin\Phi\cdot v_s+2\,\omega\cos\Phi\cdot\cos\theta\,v_z$$

$$=K_\theta-\frac{1}{\rho}\frac{\partial p}{s\partial \theta}+\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial \Theta}{s\partial \theta}+\nabla^2v_\theta-\frac{v_\theta}{s^2}+\frac{2}{s^2}\frac{\partial v_s}{\partial \theta}\right],$$

$$\frac{dv_z}{dt}-2\,\omega\cos\Phi\,(\cos\theta\cdot v_\theta+\sin\theta\cdot v_s)$$

$$=K_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial \Theta}{\partial z}+\nabla^2v_z\right],$$

where

$$\Theta=\frac{1}{s}\frac{\partial}{\partial s}(sv_s)+\frac{1}{s}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z},$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_s\frac{\partial}{\partial s}+\frac{v_\theta}{s}\frac{\partial}{\partial\theta}+v_z\frac{\partial}{\partial z}.$$

If r, θ, λ are spherical polar coordinates, where θ is the colatitude and λ the longitude, we have

$$h_1=1,\quad h_2=\frac{1}{r},\quad h_3=\frac{1}{r\sin\theta},\quad \omega_r=\omega\cos\theta,\ \omega_\theta=-\omega\sin\theta,\ \omega_\lambda=0.$$

We have

$$\Theta=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial v_\lambda}{\partial\lambda},$$

and

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+v_\theta\frac{\partial}{r\partial\theta}+v_\lambda\frac{\partial}{r\sin\theta\cdot\partial\lambda},$$

and the equations of motion are

$$\frac{dv_r}{dt}-\frac{v_\theta^2+v_\lambda^2}{r}-2\omega\sin\theta\cdot v_\lambda$$
$$=K_r-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial\Theta}{\partial r}+\nabla^2v_r-\frac{2v_r}{r^2}-2\frac{\cot\theta}{r^2}v_\theta-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right.$$
$$\left.-\frac{2}{r^2\sin\theta}\frac{\partial v_\lambda}{\partial\lambda}\right],$$

$$\frac{dv_\theta}{dt}+\frac{v_r v_\theta}{r}-\frac{\cot\theta}{r}v_\lambda^2-2\omega\cos\theta\cdot v_\lambda$$
$$=K_\theta-\frac{1}{\rho}\frac{\partial p}{r\partial\theta}+\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial\Theta}{r\partial\theta}+\nabla^2v_\theta-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\lambda}{\partial\lambda}\right.$$
$$\left.+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right],$$

$$\frac{dv_\lambda}{dt}+\frac{v_r v_\lambda}{r}+\frac{\cot\theta}{r}v_\theta v_\lambda+2\omega(\cos\theta\cdot v_\theta+\sin\theta\cdot v_r)$$
$$=K_\lambda-\frac{1}{\rho\cdot r\sin\theta}\frac{\partial p}{\partial\lambda}+\frac{\mu}{\rho}\left[\frac{1}{3}\frac{\partial\Theta}{r\sin\theta\partial\lambda}+\nabla^2v_\lambda-\frac{v_\lambda}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\lambda}\right.$$
$$\left.+\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial\lambda}\right].$$

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