Dynamics of the Movement of Atmospheric Vortices

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Abstract

By assuming the law of conservation of absolute angular momentum in the travelling atmospheric vortex with vertical axis, it is shown that the vortex moves with the same speed as that of the general current in which no vertical shear exists. Also in the case of a general current in which there exists the vertical shear, a formula of the movement of the vortex is obtained.

1. Introduction

Hitherto, a great many empirical laws concerning movements of such atmospheric vortices as typhoons have been known but no satisfactory dynamical study of them has been done. C, G, Rosby recently gave the dynamical relationship between the displacement of atmospheric vortices and the rate of variation of the Coriolis' parameter with latitude. In this paper we shall attempt to explain dynamically the movement of atmospheric vortices in terms of the properties of a general current.

2. Barotropic general current and movements of atmospheric vortices

We shall consider a simple vortex in which air spins around a vertical axis and pressure may be regarded as a function of r, the horizontal distance from the axis. We shall first give a general consideration. A rectangular system of coordinates x, y and z will here be introduced, with x increasing eastward, y northward and z vertically upward. If the corresponding velocity components are denoted by u, v and w, density by ρ, pressure by P, angular velocity of the earth's rotation by ω, and geographical latitude by φ, the Eulerian equations of motion become

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}-\lambda v+lw=-\dfrac{1}{\rho}\dfrac{\partial P}{\partial x} \\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}+\lambda u=-\dfrac{1}{\rho}\dfrac{\partial P}{\partial y} \\ \dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+v\dfrac{\partial w}{\partial y}+w\dfrac{\partial w}{\partial z}-lu=-g-\dfrac{1}{\rho}\dfrac{\partial P}{\partial z} \end{cases}$$

where $\lambda=2\omega\sin\varphi$ and $l=2\omega\cos\varphi$, which will here be assumed to be constants.

The equation of continuity for an incompressible atmosphere becomes

$$(2)\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

If we denote an arbitrary system of stationary solutions satisfying the equations

(1) and (2) by

$$(3)\qquad \begin{cases} u=f(x,y,z), \quad v=g(x,y,z), \quad w=h(x,y,z) \\ P=\varphi(x,y,z) \end{cases}$$

then

$$(4)\qquad \begin{cases} f\dfrac{\partial f}{\partial x}+g\dfrac{\partial f}{\partial y}+h\dfrac{\partial f}{\partial z}-\lambda g+lh=-\dfrac{1}{\rho}\dfrac{\partial \varphi}{\partial x} \\ f\dfrac{\partial g}{\partial x}+g\dfrac{\partial g}{\partial y}+h\dfrac{\partial g}{\partial z}+\lambda f=-\dfrac{1}{\rho}\dfrac{\partial \varphi}{\partial y} \\ f\dfrac{\partial h}{\partial x}+g\dfrac{\partial h}{\partial y}+h\dfrac{\partial h}{\partial z}-lf=-g-\dfrac{1}{\rho}\dfrac{\partial \varphi}{\partial z} \\ \dfrac{\partial f}{\partial x}+\dfrac{\partial g}{\partial y}+\dfrac{\partial h}{\partial z}=0 \end{cases}$$

From (3) and (4), we can formulate a system of solutions satisfying the equations (1) and (2) as follows,

$$(5)\qquad \begin{cases} u=f(x-\mathrm{U}t,\ y,\ z)+\mathrm{U} \\ v=g(x-\mathrm{U}t,\ y,\ z) \\ w=h(x-\mathrm{U}t,\ y,\ z) \\ P=\varphi(x-\mathrm{U}t,\ y,\ z)-\rho\lambda\mathrm{U}y+\rho l\mathrm{U}z+const \end{cases}$$

where U is an arbitrary constant and $-\rho\lambda\mathrm{U}y+\rho l\mathrm{U}z+const.$ may be regarded as the pressure corresponding to the constant wind speed U of a zonal current in the west-east direction. It will be seen from (5) that a stationary system of winds (3) may move with the same speed U as that of the zonal current in the west-east direction.

If we now omit the vertical component of the Coriolis' force and the horizontal component of the the Coriolis' force involving w, also in the case where λ is not a constant but a function of y only, we shall be able to obtain a system of solutions as follows,

$$\begin{cases} u=f(x-\mathrm{U}t,\ y,\ z)+\mathrm{U} \\ v=g(x-\mathrm{U}t,\ y,\ z) \\ w=h(x-\mathrm{U}t,\ y,\ z) \\ P=\varphi(x-\mathrm{U}t,\ y,\ z)-\rho\mathrm{U}\displaystyle\int^{y}\lambda\,dy+const. \end{cases}$$

from which we can also obtain the same result as above.

From the above general result, we can infer that a stationary vortex of atmosphere may travel with the same constant speed as that of a zonal current. But we shall next discuss this problem from a different point of view. λ will again be assumed to be constant. And the law of conservation of absolute angular mometum will be assumed to be valid, also in the travelling vortices with vertical axes corresponding to travelling circular isobaric systems, while it is well known to be valid in the staying vortices with vertical axes corresponding to staying circular isobaric systems. The Lagrangian equations of motion for the x and y components may now be expressed in the form

$$(6)\qquad \begin{cases} \dfrac{dx}{dt}=u, & \dfrac{dy}{dt}=v \\ \dfrac{du}{dt}=\lambda v-\dfrac{1}{\rho}\dfrac{\partial P}{\partial x}, & \dfrac{dv}{dt}=-\lambda u-\dfrac{1}{\rho}\dfrac{\partial P}{\partial y} \end{cases}$$

where the horizontal component of the Coriolis' force involving w is neglected. If we denote the x-and y-coordinates of the center of a vortex with vertical axis by $X(t)$ and $Y(t)$, which are functions of t, the absolute angular momentum around the moving center will be expressed in the form

$$M=\left(v-\frac{dY}{dt}\right)(x-X)-\left(u-\frac{dX}{dt}\right)(y-Y)+\frac{\lambda}{2}\{(x-X)^2+(y-Y)^2\}$$

From this we get

$$\begin{aligned}\frac{dM}{dt}=&\left(\frac{dv}{dt}-\frac{d^2Y}{dt^2}\right)(x-X)+\left(v-\frac{dY}{dt}\right)\left(\frac{dx}{dt}-\frac{dX}{dt}\right)\\ &-\left(\frac{du}{dt}-\frac{d^2X}{dt^2}\right)(y-Y)-\left(u-\frac{dX}{dt}\right)\left(\frac{dy}{dt}-\frac{dY}{dt}\right)\\ &+\lambda(x-X)\left(\frac{dx}{dt}-\frac{dX}{dt}\right)+\lambda(y-Y)\left(\frac{dy}{dt}-\frac{dY}{dt}\right)\end{aligned}$$

If we substitute the equations (6) in the right-hand side above, we shall obtain

$$(7)\qquad \begin{aligned}\frac{dM}{dt}=&\frac{y-Y}{\rho}\frac{\partial P}{\partial x}-\frac{x-X}{\rho}\frac{\partial P}{\partial y}-\left(\lambda\frac{dX}{dt}+\frac{d^2Y}{dt^2}\right)(x-X)\\ &-\left(\lambda\frac{dY}{dt}-\frac{d^2X}{dt^2}\right)(y-Y)\end{aligned}$$

And if the law of conservation of absolute angular momentum is assumed to be valid, we shall obtain

$$\frac{dM}{dt}=0$$

that is

$$(8)\qquad \begin{aligned}&(y-Y)\frac{\partial P}{\partial x}-(x-X)\frac{\partial P}{\partial y}-\rho\left\{\left(\lambda\frac{dX}{dt}+\frac{d^2Y}{dt^2}\right)(x-X)\right.\\ &\left.+\left(\lambda\frac{dY}{dt}-\frac{d^2X}{dt^2}\right)(y-Y)\right\}=0\end{aligned}$$

If now we omit the vertical components of acceleration and Coriolis' force, we may replace the equation of motion for the z component by the hydrostatic equation

$$-g-\frac{1}{\rho}\frac{\partial P}{\partial z}=0$$

From this, (8) becomes

$$(9)\qquad (y-Y)\frac{\partial P}{\partial x}-(x-X)\frac{\partial P}{\partial y}+\frac{1}{g}\{a(x-X)+b(y-Y)\}\frac{\partial P}{\partial z}=0$$

where $a=\lambda\frac{dX}{dt}+\frac{d^2Y}{dt^2}$ and $b=\lambda\frac{dY}{dt}-\frac{d^2X}{dt^2}$. If we regard (9) as a partial differential equation for P, the characteristic equation of it will become

$$\frac{dx}{y-Y}=\frac{dy}{-(x-X)}=\frac{gdz}{a(x-X)+b(y-Y)}$$

The integrals of this equation are as follows.

$$r\equiv\sqrt{(x-X)^2+(y-Y)^2}=const.$$

$$\zeta\equiv bx-ay-gz=const.$$

The general solution of (9) will therefore become

$$P=f(r,\zeta,t)$$

where f is an arbitrary function.

From the above general solution and the hydrostatic equation, we get, assuming ρ to be constant,

$$\frac{\partial P}{\partial z}=\frac{\partial f}{\partial \zeta}\,\frac{\partial \zeta}{\partial z}=-g\frac{\partial f}{\partial \zeta}=-\rho g$$

Therefore

$$\frac{\partial f}{\partial \zeta}=\rho$$

From this, we obtain

$$f(r,\zeta,t)=\rho\zeta+\varphi(r,t)$$

that is,

$$(10)\qquad P=\varphi(r,t)+\rho\left(\lambda\frac{dY}{dt}-\frac{d^2X}{dt^2}\right)x-\rho\left(\lambda\frac{dX}{dt}+\frac{d^2Y}{dt^2}\right)y-\rho gz$$

where φ is an arbitrary function of $r=\sqrt{(x-X)^2+(y-Y)^2}$ and t. We may here regard the first term as the pressure corresponding to the vortex and the remaining terms as that corresponding to the general current. In this case, assuming the geostrophic wind, the x-and y-components of the wind speed of the general current will become

$$(11)\qquad \begin{cases} U=\frac{1}{\lambda}\frac{d^2Y}{dt^2}+\frac{dX}{dt} \\ V=-\frac{1}{\lambda}\frac{d^2X}{dt^2}+\frac{dY}{dt} \end{cases}$$

It should here be remarked that the law of conservation of absolute angular momentum around the center of a moving vortex can be derived from the Eulerian equations of motion for the pressure (10), that is

$$\begin{cases} \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}-\lambda v=-\dfrac{1}{\rho}\dfrac{\partial \varphi}{\partial x}+\dfrac{d^2X}{dt^2}-\lambda\dfrac{dY}{dt} \\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+\lambda u=-\dfrac{1}{\rho}\dfrac{\partial \varphi}{\partial y}+\dfrac{d^2Y}{dt^2}+\lambda\dfrac{dX}{dt} \end{cases}$$

where $X(t)$ and $Y(t)$ are given functions of t and φ is a function of $r=\sqrt{(x-X)^2+(y-Y)^2}$. we shall next consider the following several special cases.

i), The case of $X(t)=ct$ and $Y(t)=0$ ($c=const.$).

From the above general theory, in this case, we get

$$\begin{cases} P=\varphi(r,t)-\rho\lambda Uy-\rho gz \\ U=c, \quad V=0 \end{cases}$$

where $r=\sqrt{(x-ct)^2+y^2}$. It will be seen from above that the vortex moves with the same speed as that of the general current.

ii), The case of $X(t)=\dfrac{a\lambda}{2}t^2$ and $Y(t)=at$ ($a=const.$).

In this case, we obtain

$$\begin{cases} P=\varphi(r,t)-\rho a\lambda^2 ty-\rho gz \\ U=\lambda at, \quad V=0 \end{cases}$$

where $r=\sqrt{\left(x-\dfrac{a\lambda}{2}t^2\right)^2+(y-at)^2}$

The general current is zonal but its speed changes with time, and isobars corresponding to it are in a definite direction but the pressure gradient changes with time. And the vortex moves along a parabola $\left(X=\dfrac{a\lambda}{2}t^2,\ Y=at\right)$, crossing the isobars of the general current.

3. Baroclinic general current and movements of atmospheric vortices

In the case of a baroclinic general current in which there exists the vertical shear of the zonal wind, the above theory can not be applied. In this case, it is difficult to obtain a satisfactory theory concerning movements of vortices, but we shall here attempt to consider this problem.

Let the general current be expressed in the form

$$u=U(z), \quad v=w=0$$

From the hydrostatic eqation $\dfrac{1}{\rho}\dfrac{\partial P}{\partial z}+g=0$ and the equation of state $P=\rho RT$, we get the pressure at the height of z in the form

$$P=P_H e^{\frac{g}{R}\int_z^H \frac{dz}{T}} \tag{12}$$

If it is assumed that above the height $z=H$, no perturbation exists exept a general current, P_H may be regaded as a function of y only. If $x=x_0$ and $y=y_0$ are the coordinates of the center of a vortex, from (12) we shall obtain

$$\left(\frac{\partial P}{\partial x}\right)_0=P\frac{g}{R}\int_z^H\left(\frac{\partial}{\partial x}\left(\frac{1}{T}\right)\right)_0 dz=0 \tag{13}$$

where we denote the values at $x=x_0$ and $y=y_0$ by the subscript 0. We assume that as soon as a temperature distribution at each level moves with the speed $K(z)$, which appears to depend on the general current $U(z)$, frictional resistance and others, new pressure distribution is formed in accordance with (12), and corresponding to this new pressure distribution, the temperature distribution is moreover renewed, and so on. Thus, starting from distribution (12) at $t=t_0$, the pressure $P_{\delta t}$ at $t=t_0+\delta t$ becomes

$$P_{\delta t}=P_H e^{\frac{g}{R}\int_z^H \frac{dz}{T(x-K\delta t,y,z)}}$$

If we neglect the higher order of δt,

$$P_{\delta t}=P_H e^{\frac{g}{R}\int_z^H \{\frac{1}{T(x,y,z)}-K\delta t\frac{\partial}{\partial x}(\frac{1}{T(x,y,z)})\}dz}$$

Therefore,

$$\frac{\partial P_{\delta t}}{\partial x}=P_{\delta t}\frac{g}{R}\int_z^H\left\{\frac{\partial}{\partial x}\left(\frac{1}{T(x,y,z)}\right)-K\delta t\frac{\partial^2}{\partial x^2}\left(\frac{1}{T(x,y,z)}\right)\right\}dz \tag{14}$$

Thus, if the center of pressure at the height of z changes from (x_0, y_0) to $(x_0+\delta x, y_0)$,

$$\left(\frac{\partial P_{\delta t}}{\partial x}\right)_{\substack{x=x_0+\delta x\\ y=y_0}}=0$$

From (14), expanding in the power series of δx, we get

$$\int_z^H\Big[\Big(\frac{\partial}{\partial x}\Big(\frac{1}{T}\Big)\Big)_0+\delta x\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{T}\Big)\Big)_0+\cdots\cdots$$
$$-K\delta t\Big\{\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{T}\Big)\Big)_0+\delta x\Big(\frac{\partial^3}{\partial x^3}\Big(\frac{1}{T}\Big)\Big)_0+\cdots\cdots\Big\}\Big]dz=0$$

If we neglect the higher order of δx and δt, we shall obtain

$$\int_z^H\Big\{\Big(\frac{\partial}{\partial x}\Big(\frac{1}{T}\Big)\Big)_0+\delta x\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{T}\Big)\Big)_0-K\delta t\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{T}\Big)\Big)_0\Big\}dz=0$$

From above and (13), we get

$$\delta x\int_z^H\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{T}\Big)\Big)_0 dz-\delta t\int_z^H K\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{T}\Big)\Big)_0 dz=0$$

That is,

$$\frac{\delta x}{\delta t}=\int_z^H K\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{T}\Big)\Big)_0 dz\Big/\int_z^H\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{T}\Big)\Big)_0 dz \tag{15}$$

which represents the speed of the movement of the center at the height of z. Especially when $K(z)=U(z)$, we get

$$\frac{\delta x}{\delta t}=\int_z^H U\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{T}\Big)\Big)_0 dz\Big/\int_z^H\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{T}\Big)\Big)_0 dz \tag{16}$$

If we assume $U(z)=U_0(const.)$, we shall get

$$\frac{\delta x}{\delta t}=c$$

which agrees with the result in section 1.

In (15) or (16), the speed of the movement of the center at each level may generally be different. But, for instance, when $\left(\frac{\partial^2}{\partial x^2}\left(\frac{1}{T}\right)\right)_0 \neq 0$ and $U(z)=U_0$ for the interval $\alpha \leqq z \leqq \beta$, and $\left(\frac{\partial^2}{\partial x^2}\left(\frac{1}{T}\right)\right)=0$ and $U(z)$ is arbitrary for the other intervals, it will be seen from (16) that the speed of the movement of the cnter at all levels is equal to U_0. And we may consider that if the speed of the movement of the center at each level is different, the vortex will be destroyed.