Symmetries and Casimir invariants for perfect fluid

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Abstract. We investigate the relations between symmetries of an adiabatic inviscid fluid and Casimir invariants of Eulerian Poisson bracket. It is shown that there exist two types of inner symmetries in addition to external symmetries. Noether's conserved quantities due to these inner symmetries are Casimir invariants when they are written in terms of Eulerian fields only. We construct the most general form for the two types of such quantities that correspond to these symmetries. However, these are shown to be equivalent. The most general Casimir of the system is shown to be the well-known Casimir functional. For a zero potential vorticity fluid, helicity is also allowed for the independent Casimir.

1. Introduction

The stability analysis of the systems is an important problem in fluid mechanics. Several methods have been devised to investigate stability problems. Arnol'd (1966, 1969) introduced a powerful method based on Hamiltonian theory of fluid mechanics. Since then Arnol'd's method have been applied to various problems in hydrodynamic stability theories (see, for example, Holm et al., 1985; Abarbanel et al., 1986). In Arnol'd's theory, Casimir invariants of Poisson bracket for equations of motion play an essential role. To apply this theory, it is important to examine Casimirs as general as possible for a given system.

For a three-dimensional adiabatic inviscid fluid, Poisson bracket which is written only in terms of Eulerian variables is discovered (Morrison et al., 1980, 1982). This bracket has Casimir invariants. It is known that the singular structure of this bracket is related to the projection from Lagrangian to Eulerian description (Holm et al., 1983; Marsden et al., 1984; Salmon, 1988). However, we do not know the relations between Casimir invariants and fluid properties sufficiently well, and then we cannot know how to find the most general Casimir invariants.

Some symmetries in the Lagrangian description will be "unvisible symmetries" in the Eulerian description, and then conserved quantities with respect to these symmetries will be "unvisible quantities" (i.e. Casimir invariants) for the Eulerian Poisson bracket. So, in this paper, we try to examine relations between symmetries of a three-dimensional adiabatic inviscid fluid in the Lagrangian description, and Casimir invariants of the Eulerian Poisson bracket of this fluid. We find there are two types of inner symmetries besides space-time symmetries. From these inner symmetries, we obtain generalized vorticity and specific entropy as locally conserved quantities. Moreover, we find that globally conserved quantities which are obtained from inner symmetries are generally Casimirs when they are written in terms of Eulerian variables only. We construct the most general form for such Casimirs. They essentially coincide with the well-known Casimir which is the integral of the product of density and arbitrary function of potential vorticity and specific entropy. For the zero potential vorticity fluid, helicity is also allowed for other forms of the Casimir besides the general one.
2. The model

Hamilton's principle for fluid takes the most natural form when we describe fluid motion from the Lagrangian point of view. The action can be obtained straightforwardly by generalizing the usual particle mechanics action to the continuous case. To make the system conserve mass automatically, it is convenient to restrict the labeling coordinate \( a^i \) \((i = 1, 2, 3)\) so that
\[
d(\text{mass}) = d^3a
\]  
holds. We must also make the fluid be adiabatic. Considering these restrictions, we will be able to start with the action of the form
\[
I = \int \int \int d^3a \, d\tau \left[ \frac{1}{2} \left( \frac{\partial x^p}{\partial \tau} \right)^2 - E(\rho, S) - \Phi(x) + \rho \frac{\partial (x)}{\partial(a)} - \rho^{-1} \right] + \eta \frac{\partial S}{\partial \tau}.
\]  
(2.2)

Here \( x^p \) \((p = 1, 2, 3)\), \( S \), and \( \rho \) represent position, specific entropy and density of a fluid particle, respectively, and \( \partial(x)/\partial(a) \) represents the Jacobian. The internal energy \( E \) (potential energy \( \Phi \) is assumed to be a prescribed function of density \( \rho \) and specific entropy \( S \) (position \( x \)). The last two terms are added to restrict the labeling coordinate to satisfy condition (2.1) and to satisfy the adiabatic condition. Here \( p \) and \( \eta \) are Lagrangian multiplier fields. We assume \( x^p \), \( S \), \( \rho \), \( p \) and \( \eta \) to be independent scalar fields on the labeling space, i.e. \( S'(a^i, \tau') = S(a, \tau), \) etc., hold if \( a \) and \( a^i \) give the same fluid particle and \( \tau \) and \( \tau' \) give the same time. Hamilton's principle with respect to \( x^p \) \((p = 1, 2, 3)\), \( S \), \( \rho \) and \( \eta \) yields
\[
\delta x^p: \quad \frac{\partial^2 x^p}{\partial \tau^2} = -\frac{\partial (x)}{\partial (a)} \frac{\partial \rho}{\partial x^p} - \frac{\partial \Phi}{\partial x^p},
\]  
(2.3)

\[
\delta S: \quad T = -\frac{\partial \eta}{\partial \tau},
\]  
(2.4)

\[
\delta \rho: \quad p = \rho^2 \frac{\partial E}{\partial \rho},
\]  
(2.5)

\[
\delta p: \quad \frac{\partial (x)}{\partial (a)} = \rho^{-1},
\]  
(2.6)

\[
\delta \eta: \quad \frac{\partial S}{\partial \tau} = 0,
\]  
(2.7)

where we define temperature \( T \) so that the thermodynamic relation holds, i.e.
\[
T = \frac{\partial E}{\partial S}.
\]  
(2.8)
Eq. (2.3) is just the equation of motion for an inviscid fluid when eq. (2.6) is substituted. From eq. (2.5), we can regard the field \( p \) as pressure of the fluid. Eq. (2.7) guarantees that the specific entropy is constant along fluid particles. Eq. (2.4) gives the relation between the auxiliary field \( \eta \) and the physical quantities. The mass conservation obviously holds because of condition (2.1). In fact, we can obtain the usual mass conservation law simply by operating \( \partial / \partial \tau \) on eq. (2.6), that is,

\[
\frac{\partial p}{\partial \tau} + \rho \frac{\partial u^p}{\partial x^p} = 0,
\]

where \( u^p = \partial x^p / \partial \tau \) is a velocity. As eqs. (2.5) and (2.6) simply give the non-dynamic relations of the fields \( p \) and \( \rho \) to position \( x^p \), we can regard \( \rho \) and \( p \) as dependent fields of \( x^p \) for realistic motion. Then the independent dynamic fields we must consider are only \( x^p \), \( S \) and \( \eta \).

3. Noether's currents

Next, we examine all symmetries of the model action (2.2). As is briefly shown in appendix A, we have a conserved current (called Noether's current) if there exists a transformation which keeps the action unchanged (or changes the action into the divergent form). First, let us examine the transformations in \( a \) and \( \tau \) that make the conservation law. They are given by (see appendix B)

\[
\delta a' = \epsilon \xi' (a), \quad \delta \tau = \epsilon' = \text{constant},
\]

with the condition

\[
\frac{\partial \xi'}{\partial a'} = 0.
\]

Here repeated Latin indices are summed from 1 to 3. If we use Noether's theorem on the symmetry \( \delta \tau = \epsilon' \), \( \delta a = 0 \), we have the energy conservation law

\[
\frac{\partial}{\partial \tau} \left( \frac{1}{2} \mathbf{u}^2 + E + \Phi \right) + \frac{\partial}{\partial \alpha} \left( p a' \frac{\partial x^p}{\partial \tau} \right) = 0,
\]

with \( a' = \rho^{-1} \partial (\partial x^p / \partial a) \) (cofactor of the matrix \( \partial x^p / \partial a' \)). Eq. (3.4) can easily be rewritten into the well-known form in \((x, t)\) space as

\[
\frac{\partial}{\partial t} \left[ \rho \left( \frac{1}{2} \mathbf{u}^2 + E + \Phi \right) \right] + \frac{\partial}{\partial x^p} \left( \rho \left( \frac{1}{2} \mathbf{u}^2 + E + \Phi \right) + p \right) \mathbf{u}^p = 0.
\]

Hereafter we use overbars to denote quantities that are functions of position \( x \) and time \( t \).

Similarly, from the symmetry \( \delta \tau = 0 \) and \( \delta a' = \epsilon \xi' \), we obtain Noether's currents due to mass conserved relabeling symmetries. By taking \( x^p \) etc. as \( \psi_a \) and setting \( \delta a^p = \epsilon \xi^a \xi' \), \( \delta^p x^p = -\epsilon \xi' \partial x^p / \partial a' \) and \( \delta^i S = -\epsilon \xi' \partial S / \partial a' \) in eq. (A.2), we have

\[
N^0 = \epsilon \xi' \left( \frac{\partial x^p}{\partial \tau} \cdot \frac{\partial \mathbf{u}}{\partial \alpha'} + \eta \frac{\partial S}{\partial a'} \right) = \epsilon \xi'/J,
\]

\[
N^i = \epsilon \xi' \left[ -\frac{1}{2} \left( \frac{\partial \mathbf{x}}{\partial \tau} \right)^2 + \frac{\rho}{\rho} + E + \Phi \right] = \epsilon \xi'/U
\]

for Noether's currents. Though there exist infinite numbers of Noether's currents due to the infinite degrees of freedom of \( \delta a' \), conservation equations from these currents are not
independent. From eqs. (3.1) and (3.3), the conservation equation can be rewritten as follows:

$$\xi^i \left( \frac{\partial A_i}{\partial \tau} + \frac{\partial U}{\partial \alpha^i} \right) = 0. \tag{3.8}$$

As $\xi^i$ is an arbitrary function of $\alpha$ except under the condition (3.3), we can obtain the relation

$$\frac{\partial A_i}{\partial \tau} + \frac{\partial U}{\partial \alpha^i} = 0. \tag{3.9}$$

This equation was first obtained by Eckart (1960) using the energy–momentum method. From eq. (3.9), we obtain one strong conservation law:

$$\frac{\partial V^k}{\partial \tau} = 0, \tag{3.10}$$

with

$$V^k = \varepsilon^{ijk} \frac{\partial A_j}{\partial \alpha^i}. \tag{3.11}$$

This is the most general local conservation law for vorticity. (For the special case, this equation was derived first by Ripa (1981) and Salamon (1982).) Note that $V^k$ has only two independent components as it is divergence-free. From this conservation law, we can construct a locally conserved Eulerian quantity. If we project $V^k$ onto the $\tau$-independent vector $\partial S/\partial \alpha^k$, we have the conserved quantity $q$ with

$$q = V^k \frac{\partial S}{\partial \alpha^k} = \varepsilon^{ijk} \left( \frac{\partial u^j}{\partial \alpha^i} \cdot \frac{\partial x^p}{\partial \alpha^i} + \frac{\partial \eta}{\partial \alpha^i} \cdot \frac{\partial S}{\partial \alpha^k} \right) \cdot \frac{\partial S}{\partial \alpha^k}$$

$$= \frac{\partial (x)}{\partial (a)} \varepsilon^{pqr} \frac{\partial \bar{u}^r}{\partial x^q} \cdot \frac{\partial S}{\partial x^r} = \frac{1}{\rho} \left( \nabla \times \bar{u} \right) \cdot \nabla S. \tag{3.12}$$

This conservation law is known as Ertel's theorem (1942), and the conserved quantity is the potential vorticity. Similarly, the quantity obtained by projecting $V^k$ onto $\partial q/\partial \alpha^k$, etc., is also conserved. However, it contains the Lagrangian field $\eta$. It is easy to see that conserved quantities which are derived from $V^k$ and do not contain $\eta$ are $q$ (and its product with an arbitrary function of $S$) only.

We can also rewrite the conservation law (3.10) into another Eulerian form. If we integrate $V^k$ on a certain two-dimensional region $\Sigma$ on labeling space, the integrated quantity is also conserved. It can be rewritten as follows:

$$\Gamma = \int_{\Sigma} V^k \, d\alpha_k = \int_{\Sigma} \varepsilon^{ijk} \frac{\partial A_j}{\partial \alpha^i} \, d\sigma_k$$

$$= - \int_{\partial \Sigma} A_k \, d\alpha^k = - \int_{\partial \Sigma} (\bar{u} + \eta \nabla S) \cdot dx,$$ \tag{3.13}

where Stokes' theorem is used. When $\partial \Sigma$ is taken on the constant $S$ surface, we get the well-known circulation conservation law for Kelvin's theorem (Eckart, 1960).

The action (2.2) has more symmetries. If the external potential $\Phi$ is translationally (rotationally) invariant, we have linear (angular) momentum conservation as Noether's conservation law. It can be obtained by setting $\delta \alpha^p = 0$, $\delta x^p = \delta x^p = \varepsilon^p = (\varepsilon^{pqr} x^q x^r)$ in eq. (A.2), and we have

$$\frac{\partial}{\partial \tau} (u^p \delta x^p) + \frac{\partial}{\partial \alpha^i} (p \alpha^i \cdot \delta x^p) = 0. \tag{3.14}$$
This equation can be easily rewritten, into the form in physical space, as
\[
\frac{\partial}{\partial t} \left( \tilde{\rho} \tilde{u}^q \delta x^q \right) + \frac{\partial}{\partial x^p} \left[ \left( \tilde{\rho} \tilde{u}^p \tilde{u}^q + p \delta^{pq} \right) \delta x^q \right] = 0.
\] (3.15)

Similarly, if the external potential \( \Phi \) has translational symmetry in the \( x^p \) direction, we have the generalized Noether conservation law for the infinitesimal Galilean transformation with \( \delta a^a = 0 \) and \( \delta x^p = \delta x^p - \epsilon \tau \). It is given by
\[
\frac{\partial}{\partial \tau} (u^p \tau - x^p) + \frac{\partial}{\partial a^p}(p \tau \cdot a^p) = 0.
\] (3.16)

This equation is also easily rewritten into the form of physical space as
\[
\frac{\partial}{\partial t} \left[ \tilde{\rho} (\tilde{u}^p \tau - x^p) \right] + \frac{\partial}{\partial x^p} \left[ \tilde{\rho} (\tilde{u}^{p} \tau - x^{p}) \tilde{u}^p + pt \delta^{pq} \right] = 0.
\] (3.17)

This equation implies that the center-of-mass of the system must move with constant velocity.

Moreover, the action has a symmetry with \( \delta a^a = 0, \delta \eta = \epsilon k(a) \). Here \( k(a) \) is an arbitrary small function of labeling coordinate \( a \) only. Generalized Noether’s currents due to this symmetry are given by
\[
N^0 = \epsilon \delta k(a),
\] (3.18)
\[
N^1 = 0.
\] (3.19)

Conservation equations from these generalized Noether’s currents can be synthesized to one conservation law in the same manner as cases (3.6) and (3.7). The equation is just eq. (2.7), i.e. specific entropy conservation is obtained. This conservation is also regarded as due to inner symmetry because it does not relate to the space-time symmetry.

Any transformation for fields \( z^A \) can be written in the form \( \delta z^A = \epsilon \Lambda^A(z^A, a^a) \). Direct examination shows there are no more symmetries in this system except for the special choice of potentials. Then all conservation laws due to symmetries for the system have been shown.

4. Globally conserved quantities

Now, let us examine globally conserved quantities due to the inner symmetries of the system. From eqs. (3.6) and (3.7), the quantity
\[
Q = \int \int \int \text{d}^3a \cdot A \cdot \xi'(a)
\] (4.1)
is globally conserved if the surface integral of \( U \xi' \) at the boundary vanishes. Note that the locally conserved quantity (3.11) can be treated as the special case of the globally conserved quantities (4.1) with
\[
\xi' = n_k \epsilon^{ijk} \frac{\partial}{\partial a^j} \delta(a - a_0).
\] (4.2)

Here \( n_k \) is a constant and \( a_0 \) is an arbitrary inner point in labeling space. We can construct such infinitely many globally conserved quantities by choosing \( \xi' \) to be functions of \( a^i, V^i, S, \partial V^{ap}/\partial a^p \), etc., and to satisfy eq. (3.3). For example, if we choose \( n' = (n_k \epsilon^{ijk} a^j) \) as \( \xi' \), we obtain “label momentum” (“label angular momentum”) for the globally conserved quantity. This quantity is known to relate to the usual pseudo-momentum (pseudo-angular-momentum) if the labeling coordinate can be expressed in terms of the reference state (Ripa, 1981). However, generally, the quantity (4.1) contains quantities related to the Lagrangian description such as \( a^i \), and we cannot attach physical meaning to such a quantity. We obtain physically meaningful
quantities only for the special choice of $\xi^i$. We have obtained such concrete examples, which can be written in terms of Eulerian fields only.

First, if we adopt $V^i$ itself as $\xi^i$, we have

$$h = \int \int \int d^3a \, V^i A_i$$

$$= -\int \int \int d^3x \left[ (\nabla \times \bar{u}) \cdot \bar{u} + \bar{u} \cdot (\nabla \bar{\eta} \times \nabla \bar{S}) + \bar{\eta} (\nabla \times \bar{u}) \cdot \nabla \bar{S} \right].$$

(4.3)

for the conserved quantity. When $\bar{S}$ is constant, only the first term in (4.3) survives, and we obtain the conservation of helicity (Moffatt, 1969).

In general, except in the case of helicity, we can express $\xi^i$ as $\epsilon^{ijk} \partial T_k / \partial a^j$ with

$$T_k = f(q, S) \frac{\partial S_k}{\partial a^k} + g(q, S) \frac{\partial q}{\partial a^k},$$

(4.4)

where $f$ and $g$ are arbitrary functions of $q$ and $S$ only (see appendix C). From this choice of $\xi^i$, we have

$$Y = \int \int \int d^3x \left[ e(q, S) \bar{u} \cdot (\nabla \bar{q} \times \nabla \bar{S}) \right]$$

(4.5)

for a conserved Eulerian quantity, where the function $e$ is defined by $\partial f / \partial \bar{q} - \partial g / \partial \bar{S}$.

Similarly, from eqs. (3.18) and (3.19), the quantity

$$Q_\eta = \int \int \int d^3a \bar{S} k(a)$$

(4.6)

is conserved without any special condition. The most general Eulerian form for (4.6) can be obtained simply by replacing $k(a)$ by $K(q, S)$, where $K$ is an arbitrary function of potential vorticity $q$ and entropy $S$. If we redefine $SK(q, S)$ as $K(q, S)$, this quantity turns out to be

$$Z = \int \int \int d^3x \bar{\eta} K(\bar{q}, \bar{S}).$$

(4.7)

5. Canonical formalism

We can construct the canonical theory by defining the Poisson bracket. It can be defined by generalizing the usual particle mechanics bracket to fields. As the canonical momenta for $x^\rho$ and $\bar{S}$ are given by $u^\rho$ and $\bar{\eta}$, respectively, from (2.2), a natural definition for the Lagrangian Poisson bracket for arbitrary functional $F$ and $G$ is given by

$$\{ F, G \} = \int \int \int d^3a \left[ \frac{\delta F}{\delta x^\rho} \frac{\delta G}{\delta u^\rho} - \frac{\delta F}{\delta u^\rho} \frac{\delta G}{\delta x^\rho} + \frac{\delta F}{\delta \bar{\eta}} \frac{\delta G}{\delta \bar{S}} - \frac{\delta F}{\delta \bar{S}} \frac{\delta G}{\delta \bar{\eta}} \right].$$

(5.1)

Here the notation $\delta F / \delta \chi$ represents the functional derivative of $F$ with respect to function $\chi$, and it is defined by

$$\int \int \int d^3a \frac{\delta F[\chi]}{\delta \chi(a)} \lambda(a) = \left. \frac{d}{d\epsilon} \right|_{\epsilon \to 0} F[\chi + \epsilon \lambda].$$

(5.2)

where $\lambda$ is an arbitrary function which vanishes at the boundary. The Hamiltonian is derived from the action (2.2), and we have

$$H = \int \int \int d^3a \left[ \frac{1}{2} u^2 + E(\rho, S) + \Phi(x) \right].$$

(5.3)
where we assume \( \rho \) and \( p \) are a priori given by eqs. (2.5) and (2.6). From these Poisson brackets and the Hamiltonian, we have the canonical equation of an arbitrary functional \( F \) with
\[
\frac{\partial F}{\partial \tau} = \{ F, H \}.
\] (5.4)
If we replace \( F \) with \( x^{\rho}, u^{\rho}, S, \eta \) and \( \rho (= \partial(a)/\partial(x)) \), we obtain the same dynamic equations as we have obtained for Lagrangian formalism.

If functionals \( F \) and \( G \) depend only on some special function \( \phi_{\alpha} \) which is written in terms of \( x^{\rho}, u^{\rho}, S \) and \( \eta \) only, we will be able to rewrite the bracket as follows:
\[
\{ F, G \} = \int \int \int d^3 y \frac{\delta F}{\delta \phi_{\alpha}} \frac{\delta G}{\delta \phi_{\beta}} \{ \phi_{\alpha}(y_1), \phi_{\beta}(y_2) \}.
\] (5.5)
where we use the chain rule for functionals and repeated indices are summed. If we choose \( \phi_{\alpha} \) as velocity \( \vec{v}^{\rho} \), density \( \bar{\rho} \) and specific entropy \( \bar{S} \) which are functions of physical space coordinates \( y_i \) \( (i = 1, 2) \), this transformation is just the transformation from Lagrangian fields to Eulerian fields. The result for the Poisson bracket projected onto Eulerian fields is given by (see appendix D)
\[
\{ F, G \}_E = \int \int d^3 y \left[ \frac{\delta F}{\delta \bar{\rho}} \cdot \frac{\delta G}{\delta \bar{\rho}} + \frac{\delta F}{\delta \bar{S}} \cdot \frac{\delta G}{\delta \bar{S}} + \bar{\rho}^{-1} \left( \nabla \times \vec{v} \right) \cdot \left( \frac{\delta G}{\delta \bar{S}} \frac{\delta F}{\delta \bar{\rho}} - \frac{\delta G}{\delta \bar{\rho}} \frac{\delta F}{\delta \bar{S}} \right) \right]
\] (5.6)
where we denote the Eulerian Poisson bracket as \( \{ \cdot, \cdot \}_E \). Similarly, the Hamiltonian (5.3) is rewritten only in terms of the Eulerian field as
\[
\bar{H} = \int \int d^3 y \bar{\rho} \left[ \frac{1}{2} \vec{v}^2 + E(\bar{\rho}, \bar{S}) + \Phi(y) \right].
\] (5.7)

As the canonical equation (5.4) must hold for the arbitrary Eulerian functional \( F \) with projected Poisson bracket (5.6) and Hamiltonian (5.7), we can obtain the canonical equation for the Eulerian description with
\[
\frac{\partial F}{\partial \tau} = \{ F, \bar{H} \}_E.
\] (5.8)
Here, we have changed time coordinate \( \tau \) to \( t \) in (5.4) because the time coordinate independent of physical space coordinate \( y \) is given by \( t \). If we substitute \( \vec{v}^{\rho}, \bar{\rho} \) and \( \bar{S} \) for \( F \) in (5.8), we get equations of motion, mass conservation, and specific entropy conservation, respectively, for fluid in the Eulerian description. These results coincide with that obtained first by Morrison and Greene (1980, 1982).

6. Casimir invariants

A functional is called a Casimir invariant when the Poisson bracket with respect to it and any functional identically vanishes. The Eulerian Poisson bracket (5.6) is known to allow the existence of Casimirs. It was obtained by projecting the original functional space defining (5.1) to its subspace. However, though the original Lagrangian Poisson bracket (5.1) is defined by eight functions, the Eulerian Poisson bracket (5.6) is defined by only five functions. The remaining functional degrees of freedom will be left as the cause of the existence of Casimir
invariants in the projected bracket. In fact, in the case where there are only finite canonical variables, it is known that there exist \( 2n - r \) Casimirs if a \( 2n \)-dimensional phase space is projected onto \( r \)-dimensional space (Littlejohn, 1982). However, as a functional derivative corresponds to derivatives with respect to infinite numbers of variables, this theorem cannot apply in this case. We would like to find the most general Casimirs, and clarify the relations between Casimirs and the functional degrees of freedom which are left by projecting the Poisson bracket in the following.

As the Poisson bracket with respect to any Casimir and Hamiltonian must vanish by the definition of Casimir, Casimirs are also conserved quantities. On the other hand, it is easy to see that any conserved quantities in the Eulerian description must also be conserved in the Lagrangian description. (Of course, the reverse is not necessarily true.) Then, the most general Casimirs of the Eulerian Poisson bracket must exist among the family of Lagrangian conserved quantities which are written by Eulerian fields only. We have already examined all such conserved quantities in sections 3 and 4. Among them, the most likely candidates for Casimirs are conserved quantities due to inner symmetries in section 4, because such symmetries cannot be observed by Eulerian description. In the following, we shall show that this expectation is true.

For the conserved quantities due to the relabeling symmetries, we can obtain the following equations from the definition of the Poisson bracket:

\[
\{ z^a, Q \} = \xi' \frac{\partial z^a}{\partial a'} = -\Delta' z^a, \tag{6.1}
\]

where we denote Lagrangian fields \( x^a, u, S \) and \( \eta \) as \( z^a \), and the quantities \( Q \) and \( \xi' \) are given by (4.1). (Note that this equation shows that \( Q \) is the generator of the label redefinition.) So, for a functional \( F \) which is written only in terms of \( z^a \), we obtain

\[
\{ F, Q \} = \int \int \int d^3a \frac{\delta F}{\delta z^a} \Delta' z^a = -\Delta' F. \tag{6.2}
\]

If a functional \( F \) depends only on some special function \( \phi_a(y) \) which is written in terms of \( z^a \) only, we would be able to rewrite eq. (6.2) as

\[
\{ F, Q \} = \int \int \int \int d^3a \ d^3y \xi' \frac{\delta F}{\delta \phi_a} \frac{\delta z^a}{\delta \phi_a}. \tag{6.3}
\]

If we choose Eulerian fields \( \tilde{u}^a, \tilde{p} \) and \( \tilde{S} \) as \( \phi_a \), we can find the equation identically vanishing by using the results in appendix D. On the other hand, the left-hand side of eq. (6.3) can be rewritten as \( \{ F, Q \} \) if \( F \) and \( Q \) can be written in terms of Eulerian fields only. Then the result shows that the quantity \( Q \) is Casimir to the Eulerian Poisson bracket if it is written in terms of Eulerian fields only. We have found in section 4 that helicity (4.3) is such a conserved quantity for the case \( S = 0 \), and “hyper-vorticity” (4.5) for the general case. Then, hyper-vorticity is the most general Casimir obtained from the relabeling symmetries. For helicity, if we redefine Eulerian helicity \( \tilde{h} \) by the first term in (4.3), direct calculation shows the following results:

\[
\{ \tilde{u}, \tilde{h} \} = 0, \tag{6.4}
\]

\[
\{ \tilde{p}, \tilde{h} \} = 0, \tag{6.5}
\]

\[
\{ \tilde{S}, \tilde{h} \} = -2\tilde{q}. \tag{6.6}
\]

Eqs. (6.4) and (6.6) are compatible with the result mentioned above. However, the result (6.6) is surprising because the condition for Eulerian helicity to be Casimir is released to be zero potential vorticity. Then, for the special case with zero potential vorticity fluid, we can treat Eulerian helicity as being also Casimir due to relabeling symmetry.
Next, for the symmetries due to a redefinition of the auxiliary field $\eta$, we have

$$\{ \eta, Q_\eta \} = -k(a), \quad (6.7)$$

$$\{ z^\tau, Q_\eta \} = 0, \quad (6.8)$$

where $Q_\eta$ and $k(a)$ are given by eq. (4.6), and $z^\tau$ represents Lagrangian fields except $\eta$. (Note that these equations imply $Q_\eta$ to be the generator of the transformation of $\eta$.) If a functional $F$ depends only on Eulerian fields $\phi_t(y)$, similar to eq. (6.3), we have

$$\{ F, Q_\eta \} = -\int \int \int \int d^3x \, d^3y \, k(a) \frac{\delta F}{\delta \phi_t} \frac{\delta \phi_t}{\delta \eta} \quad (6.9)$$

As the functional derivatives of Eulerian fields with respect to $\eta$ all vanish, the right-hand side of eq. (6.9) also vanishes. On the other hand, the left-hand side of eq. (6.9) can be rewritten as $(F, Q_\eta)$ if $F$ and $Q_\eta$ are both Eulerian functional. We have found in section 4 that the most general form for $Q_\eta$ to be Eulerian is “hyper-entropy” (4.7). Then the most general Casimir obtained from the symmetries of redefinition of auxiliary field is hyper-entropy. The reason why hyper-entropy is Casimir for the Eulerian Poisson bracket is easily explained because the degrees of freedom of $\eta$ are hidden in the Eulerian description as relabeling degrees of freedom.

A similar survey for energy, momentum and center-of-mass shows that they are not Casimir. Thus, Casimir invariants for the Eulerian Poisson bracket are related only to the inner symmetry of the fluid, as we expected. This is explained as follows. As is known in analytical mechanics, Noether's globally conserved quantities serve as generators of the transformation of the symmetries in Hamiltonian formalism. As the transformations for relabeling and redefinition of auxiliary field cannot be expressed on a Eulerian basis, Noether's globally conserved quantities serve as null generators in Eulerian formalism. Then these quantities must be Casimirs when they are represented by Eulerian fields only.

Next, let us examine the relation between a hyper-vorticity family and a hyper-entropy family as Casimir. These are obtained by essentially independent transformations, and so it appears that they are independent. However, this is not true, as is shown in the following. If we define function $M(\tilde{\eta}, \tilde{S})$ so that $\partial M/\partial \tilde{\eta} = \epsilon(\tilde{\eta}, \tilde{S})$ holds, hyper-vorticity can be rewritten as

$$Y = \int \int \int d^3x \, \tilde{u} \cdot [\nabla M(\tilde{\eta}, \tilde{S}) \times \nabla \tilde{S}]$$

$$= \int \int \int d^3x \, \nabla \cdot [M(\tilde{\eta}, \tilde{S}) \nabla \tilde{S} \times \tilde{u}] + \int \int \int d^3x \, \tilde{\rho} M(\tilde{\eta}, \tilde{S}), \quad (6.10)$$

where partial integration is used. The first term in eq. (6.10) can be rewritten as surface integration. As a functional derivative of any surface integral identically vanishes by the definition (5.2), the first term is an apparent Casimir. Thus the character of Casimir is not affected by removing it from $Y$. The second term, on the other hand, is the same as hyper-entropy if the arbitrary function $K(\tilde{\eta}, \tilde{S})$ is defined by $\tilde{\rho} M(\tilde{\eta}, \tilde{S})$. Then, apart from the surface integral, a hyper-vorticity family coincides with a hyper-entropy family. So, apart from the integral which can be rewritten as a surface integral, the most general Casimir is hyper-entropy. For a zero potential vorticity fluid, helicity is also allowed as the Casimir independent of hyper-entropy.

7. Discussion and remarks

We have shown that we can construct the Eulerian Poisson bracket by projecting our Lagrangian Poisson bracket onto Eulerian fields. This Eulerian bracket has the singular
property, that there exist Casimirs. These Casimirs relate to inner symmetries of the fluid in the Lagrangian description. We have two types of inner symmetries. One of them is a mass conserved relabeling symmetry, and the other is a redefinition of the auxiliary field. Each of the conservation equations for these symmetries synthesizes with one local conservation law: general vorticity or specific entropy. We can also construct locally conserved Eulerian quantities (potential vorticity and specific entropy) from them.

For globally conserved quantities with inner symmetries, these quantities are Casimirs for the Eulerian Poisson bracket when they are written in terms of Eulerian fields only. From the relabeling symmetry, we obtain hyper-vorticity (4.5) for the general case and helicity for the zero potential vorticity case. From the symmetry of the auxiliary field, we have hyper-entropy (4.7) for such Casimirs. These quantities are shown to be identical except for the apparent Casimir.

The Casimir can exist due to the reduction of degrees of freedom from eight Lagrangian fields to five Eulerian fields. Then, from the analogy of the finite degrees of freedom, it seems that the general Casimir must have three functional freedoms with three variables. There are two functional degrees of freedom (three degrees of freedom of $\xi'$ minus constraint (3.3)) for relabeling symmetry, and one for symmetry of $\eta$. These degrees of freedom are equal to those due to the reduction. Globally conserved quantities (4.1) and (4.6) inherit these degrees of freedom due to symmetries. However, when arbitrary functions of $a$ are replaced by arbitrary functions with locally conserved Eulerian quantities $\tilde{q}$ and $\tilde{S}$, functional degrees of freedom are reduced from three variables to two variables. This process means that we have labeled fluid particles by locally conserved Eulerian quantities. Of course, there must exist three such independent quantities, if we want to label fluid particles completely. For the Lagrangian description, there are three locally conserved quantities (two independent components of general vorticity and specific entropy). However, one labeling degree of freedom has been lost by the projection of Eulerian fields. In order to determine the third Lagrangian coordinate, it may seem that mass conservation can be used. However, the equation $\rho = \partial(q, S, c)/\partial(x)$ cannot be explicitly solved for $c$ by Eulerian fields only. (If we consider an incompressible fluid, for example, we have one more locally conserved quantity, and then no reduction of degrees of freedom for labeling occurs. Of course, there cannot exist more than four independent locally conserved quantities, whatever fluid is assumed.) The fact that there exist only two locally conserved Eulerian quantities means we cannot label fluid particles completely by Eulerian fields only. Further, when we construct hyper-vorticity, two functional degrees of freedom of $\xi'$ are merged to one. Moreover, the hyper-vorticity family and the hyper-entropy family are shown to be identical except for a functional which can be written as a surface integral. This functional reduction seems to occur because functional degrees of freedom are very large, and we use locally conserved quantities, which have the same symmetries as global ones, as functional arguments. Then, the most general Casimir that illustrates hyper-entropy (or hyper-vorticity) contains only one arbitrary function with two variables. If we consider an incompressible fluid, for example, we obtain the most general Casimir that is the integral of one arbitrary function with three variables $\tilde{p}$, $\tilde{q}$ and $\tilde{S}$.

We have used Noether's theorem twice for inner symmetries. First it was used to derive locally conserved quantities, and second it was used to derive globally conserved quantities. Though conservation equations to derive globally conserved quantities are not independent and locally conserved equations seem to be more important, the global form is more important in discussing Casimirs. In general, Noether's globally conserved quantities are generators of the transformation for the symmetries in the Hamiltonian formalism. Thus, for the Noether's globally conserved quantities due to inner symmetry, functional arbitrarinesses represent degrees of freedom for the inner transformation. As this type of transformation cannot be represented by a Eulerian field, they are left as Casimirs in the Eulerian Poisson bracket when
conserved quantities are represented by Eulerian fields. In contrast, locally conserved quantities are used to replace labeling coordinates by Eulerian quantities. In the system we examine, the quantities which are constant along fluid particles are accidentally related to the symmetries of the system, and then obtained by synthesizing Noether's conservation equations.

For the homentropic fluid, two locally conserved quantities in the system, potential vorticity and specific entropy, become constants. Thus, we have an apparent constant for hyper-entropy. In this case, the only meaningful Casimir is Eulerian helicity. However, helicity has no functional degrees of freedom. So, in the homentropic fluid, Casimir is strictly restricted. In other words, entropy plays a very important role for degrees of freedom of Casimirs in this theory.

We started by including thermodynamical fields as well as particle mechanics fields. This will imply that fluid particles have inner structures represented by thermodynamic fields. We must have appended auxiliary field $\eta$ to the theory to involve specific entropy conservation. Thus, $\eta$ seems also to be a thermodynamic field and to be related to inner structure of fluid particle. However $\eta$ differs from the usual thermodynamical fields because it relates to the time integral of temperature along particle motion. In this sense, $\eta$ seems to be unphysical. However, for the theory we discussed, auxiliary field $\eta$ plays a very important role. For example, we cannot define the Lagrangian Poisson bracket to include entropy, and then we cannot construct hyper-entropy if $\eta$ is absent. Thus $\eta$ is similar to a ghost field in field theory.

The theory we have developed is available only for the internal region of the fluid. For the infinite fluid, there are no problems with such a restriction. However, for the realistic fluid, there always exists a fluid boundary. Therefore, we must supply boundary conditions case by case to real problems. Though this treatment presents no problems in application, the theory is more useful if it automatically contains boundary motion or boundary conditions of fluid. Extension of our theory is left to future work. For an example of the theory to include boundary motion, see Lewis et al. (1986).

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Appendix A. Derivation of Noether's currents

In general, variation of an action by means of fields variation and coordinate transformation is written up to the leading order of the variation (see, for example, Utiyama, 1978; Goldstein, 1980):

$$\delta I = \int \int \int d^4a \left( \frac{\partial \mathcal{L}}{\partial \psi_A} - \frac{\partial}{\partial a^a} \left( \frac{\partial \mathcal{L}}{\partial (\partial \psi_A / \partial a^a)} \right) \right) \delta^4 \psi_A - \frac{\partial N^u}{\partial a^u},$$

where $\mathcal{L}$ is a Lagrangian density and Noether's current $N^u$ is given by

$$N^u = -\mathcal{L} \delta a^u - \frac{\partial \mathcal{L}}{\partial (\partial \psi_A / \partial a^u)} \delta^4 \psi_A.$$  \hspace{1cm} (A.2)

We denote a field and its variations by $\psi_A$ and $\delta \psi_A$, and Lie variation $\delta^4 \psi_A$ is defined by

$$\delta^4 \psi_A = \psi_A(a) - \psi_A(a) = \delta \psi_A - \delta a^a \frac{\partial \psi_A}{\partial a^a}.$$  \hspace{1cm} (A.3)
where the arguments on $\psi_\lambda$ and $\psi_\lambda$ have the same value but the positions in space-time are different. We denote the coordinate as $a^\mu$; it represents time for $\mu = 0$, and space-like components for $\mu = i$ (1 to 3). We also denote kinds of fields by suffix $A$, and we use the convention in which repeated indices are summed up to its available number. If we apply the Euler–Lagrange equation with respect to fields $\psi_A$ to (A.1), the first term in eq. (2.15) vanishes and we obtain the equation

$$8I = - \int \int \int d^4a \frac{\partial N^n}{\partial a^n}.$$  \hspace{1cm} (A.4)

If a variation identically satisfies $\delta I = 0$ in any domain $\Omega$, the integrand itself must vanish, and we obtain the local conservation equation

$$\frac{\partial N^n}{\partial a^n} = 0.$$ \hspace{1cm} (A.5)

If a variation can be written in the form

$$8I = \int \int \int d^4a \frac{\partial F^n}{\partial a^n},$$ \hspace{1cm} (A.6)

on the other hand, we have the conservation law

$$\frac{\partial N^n}{\partial a^n} = 0.$$ \hspace{1cm} (A.7)

Here the generalized Noether's current $N^{\nu'}$ is defined as

$$N^{\nu'} = N^\rho + F^n.$$ \hspace{1cm} (A.8)

The equation (A.5) or (A.7) is just Noether's conservation law.

Appendix B. Derivation of conditions (3.1) to (3.3)

Let us examine the transformation of the coordinates to make the action (2.2) unchanged or divergence form. The transformations we must examine are the following:

$$a' = a + \delta a(a, \tau).$$ \hspace{1cm} (B.1)

$$\tau' = \tau + \delta \tau(a, \tau).$$ \hspace{1cm} (B.2)

The variation of the action by these transformations is given by

$$\delta I = \int \int \int d^4a \, d\tau \left( \delta \mathcal{L} + \frac{\partial}{\partial a'^\nu} \delta a^{\nu'} \mathcal{L} \right).$$ \hspace{1cm} (B.3)

where $\mathcal{L}$ is a Lagrangian density, i.e. the integrand of the action (2.2), and repeated Greek indices must be summed from 0 to 3. As we assumed fields $x, S, \rho, p$ and $\eta$ (hereafter we denoted these fields as $z^i$) to be scalar on $(a, \tau)$ space, $\delta z^i = 0$ is maintained. From this condition, variations of the fields are given by

$$\delta \frac{\partial z^i}{\partial a^\nu} = - \frac{\partial z^i}{\partial a^\nu} \frac{\partial a^\lambda}{\partial a^{\nu'}}.$$ \hspace{1cm} (B.4)

Then $\delta \mathcal{L}$ in eq. (B.3) can be rewritten as

$$\delta \mathcal{L} = - \frac{\partial x^\rho}{\partial \tau} \left( \frac{\partial x^\rho}{\partial a^\mu} \frac{\partial \delta a^\mu}{\partial \tau} \right) - 2p \left( \frac{\partial x^\rho}{\partial a^\mu} - a^\rho \frac{\partial x^\rho}{\partial \tau} - \frac{\partial a^\mu}{\partial \tau} \right) - \eta \frac{\partial S}{\partial a^\mu} \frac{\partial \delta a^\mu}{\partial \tau},$$ \hspace{1cm} (B.5)

where $a^\rho = \partial a^\rho / \partial (\partial x^\rho / \partial a^\rho)$. The condition $\delta I = 0$ is identically satisfied if the integrand of
eq. (B.3) identically vanishes. It is easy to see that this condition is equivalent to the following conditions:

\[
\frac{\partial \delta a^a}{\partial \tau} = 0, \quad \frac{\partial \delta \tau}{\partial a^a} = 0, \quad \frac{\partial \delta a^a}{\partial a^a} = 0. \tag{B.6}
\]

From these equations, we finally obtained eqs. (3.1) to (3.3). Further, there is no transformation which makes the action into a non-zero divergence form.

Appendix C. Proof that \( T_k \) is given by eq. (4.4)

In order for the quantity \( Q \) in eq. (4.1) to be written by Eulerian fields only, it must not change its value with the transformation of the labeling coordinate. The quantity \( A_i \) transforms as a covariant vector which transforms as

\[
A'_i = \frac{\partial a'}{\partial a^a}A^a, \tag{C.1}
\]

under the coordinate transformation \( a'' = a''(a) \). So from the invariance of quantity \( Q \) under the infinitesimal transformation of \( a \), we obtain that \( \xi^i \) must transform as a contravariant vector or contravariant vector density, and also should satisfy the condition (3.3).

The labeling coordinate is one of the curvilinear coordinates in physical space. Generally, the line element of the physical space which is represented by general curvilinear coordinate \( b' \) is given by

\[
\text{d}s^2 = \delta_{p'q'} \, \text{d}x^p \, \text{d}x^q = \frac{\partial x^p}{\partial b'^i} \frac{\partial x^q}{\partial b'^j} \, \text{d}b'^i \, \text{d}b'^j. \tag{C.2}
\]

Thus, we can define the metric in this space by

\[
\gamma_{ij} = \frac{\partial x^p}{\partial b'^i} \frac{\partial x^p}{\partial b'^j}. \tag{C.3}
\]

Note that the fundamental scalar density \( \sqrt{\gamma} \) (\( \gamma = \det(\gamma_{ij}) \)) is given by the Jacobian \( \partial(x)/\partial(a) \), and thus it is equal to \( \rho^{-1} \), when coordinate \( b' \) is chosen as labeling coordinate \( a \). We introduce the Riemannian connection in this space. Thus, covariant derivatives are defined by using coefficients of connection \( \Gamma_{ij}^k \) in the usual way (see Utiyama, 1978). We denote the covariant differential as \( \nabla_j \). Note that Eddington’s symbol \( \epsilon^{ijk} \) \( (\epsilon_{ijk}) \) can be regarded as contravariant (covariant) tensor density of weight one (minus one) for general coordinate transformation of \( b' \).

Eulerian fields \( \vec{u}^p, \vec{S} \) can be regarded as covariant quantities in the general curvilinear coordinate \( u^\rho \nabla_j x^\rho \), \( \vec{S} \) evaluated in Cartesian coordinate \( x \), and \( \rho^{-1} \) is equal to the fundamental scalar density \( \sqrt{\gamma} \) when \( b \) is chosen as the labeling coordinate. Further, derivatives with respect to \( x^\rho \) can be regarded as the covariant derivatives evaluated in Cartesian coordinates, and volume element \( \text{d}^3x \) can be rewritten as \( (\sqrt{\gamma})^{-1}\text{d}^3b \). So, any Eulerian quantities can be written in covariant form in labeling coordinates. As elemental locally conserved quantities are \( V^k \) and \( S \) only, \( \xi^i \) must be constructed by them, its derivatives, fundamental tensors and their combinations only. (Of course \( V^k \) has covariant form as \( V^k = \epsilon^{ijk} \nabla_j A_i \), \( \tag{C.4} \))
and then it transforms as contravariant vector density.) If $\xi'$ contains arbitrary functions, their arguments themselves must be rewritten in Eulerian form because argument fields must transform by themselves. However, there are only two such quantities, $q$ and $S$. Thus the general form for $\xi'$ is given by

$$\xi' = \epsilon^{ij} \nabla_j T_i,$$

with

$$T_i = C_i(q, S, \nabla q, \nabla S),$$

where we consider condition (3.3), and $C_i$ is a covariant vector function of $q$, $S$ and their derivatives. Note that covariant derivatives of tensors higher than order one are not available, because $T^1_{jk}$, as well as being metric, is not independent of time. Using covariance of $\xi'$ and $A_i$, we have the form of $Q$ in $x$ space as

$$Q = \int \int \int d^3x \epsilon^{pq} \frac{\partial C_i}{\partial x^q} \left( \bar{u}^p + \eta \frac{\partial S}{\partial x^p} \right).$$

As $Q$ must not contain $\eta$, the condition

$$\epsilon^{pq} \frac{\partial C_i}{\partial x^q} \frac{\partial S}{\partial x^p} = 0$$

must hold identically. Thus $C_i$ is easily determined as the form (4.4).

Appendix D. Derivation of eq. (5.6) from definition (5.1)

We can express Eulerian fields by means of Lagrangian fields as follows:

$$\bar{u}^p(y) = \int \int \int d^3a \delta(y - x(a)) \frac{\partial(x)}{\partial(a)} u^p(a),$$

$$\bar{\rho}(y) = \int \int \int d^3a \delta(y - x(a)),$$

$$\bar{S}(y) = \int \int \int d^3a \delta(y - x(a)) \frac{\partial(x)}{\partial(a)} S(a),$$

where $\delta(\cdots)$ represents delta function. From these equations, we obtain functional derivatives of Eulerian fields with respect to Lagrangian fields as

$$\frac{\delta \bar{u}^p(y)}{\delta x^q(a)} = -\frac{1}{\rho} \delta(y - x(a)) \frac{\partial}{\partial y^q},$$

$$\frac{\delta \bar{\rho}(y)}{\delta u^q(a)} = \frac{1}{\rho} \delta(y - x(a)) \delta u^q,$$

$$\frac{\delta \bar{S}(y)}{\delta x^p(a)} = \frac{\partial}{\partial x^p} \delta(y - x(a)),$$

$$\frac{\delta \bar{S}(y)}{\delta \bar{S}(a)} = \frac{\partial}{\partial \bar{S}} \delta(y - x(a)),$$

and others are zero. Substituting these results into eq. (5.1) and then into eq. (5.5), we finally obtain eq. (5.6).
References


