Figure S1. Transfer functions for extracting medium-scale (red), synoptic (black), and stationary (blue) waves. The ordinate represents transfer rates, and the abscissa represents period of time with unit of days.
Figure S2. Meridional cross sections of total Eulerian mass stream function averaged during winter (December to March) driven by (a) all eddies, (b) all friction, and (c) all diabatic heating without inclusion of non-linear terms; (d) all forcings including non-linear terms; and (e) all eddies, (f) all friction, and (g) all diabatic heating with inclusion of non-linear terms. Panel (h) shows the Eulerian mass stream function calculated directly from the observed monthly-mean meridional and vertical winds. Areas shown are from 20°S to the north pole. Contour interval is $1 \times 10^{10}$ kg s$^{-1}$; solid lines indicate zero and positive values, dashed lines indicate negative values. Light, medium, and dark shading indicate month-to-month standard deviations of deseasonalized variation of $1 \times 10^{10}$, $3 \times 10^{10}$, and $5 \times 10^{10}$ kg s$^{-1}$, respectively.
The energy equation for specific waves (such as synoptic waves) is obtained as follows.

Let us define a Lanczos filter to extract a specific frequency range by

\[ F \cdot u(t) \equiv \int_{-T}^{T} ds G(s) u(t + s), \]  

(S3.1)

where \( G(s) \) is a filtering function to extract some specific spectrum range, say “1”.

Note that the operator \( F \) is exchangeable with the time derivative and spatial operation such as \( \nabla \cdot \). In the following, we substitute \( u(t)_1 \equiv F \cdot u(t) \) for simplicity.

When \( F \) is applied to the governing equations of disturbance, we obtain the governing equation of a wave component with a frequency range of “1”. For example, the zonal wind equation becomes

\[
\begin{aligned}
\frac{\partial u'_i}{\partial t} &+ \frac{[\bar{u} - \bar{u}]_i}{a \cos \phi} + \frac{[v \cdot \partial u'_i]}{a \cdot \partial \phi} + \frac{[\bar{w} \cdot \partial u'_i]}{\partial z} - f v' \iota \\
&= -\frac{\tan \phi}{a} \{[\bar{u} v']_i + [u' \bar{v}]_i\} + \frac{[v' \cdot \partial u]}{a \cdot \partial \phi} + \frac{[w' \cdot \partial u]}{\partial z} + \frac{1}{a \cos \phi} \frac{\partial \Phi'}{\partial \lambda} \\
&= X'_i - \frac{1}{a \cos \phi} \frac{\partial}{\partial \lambda} \{[(u' v')_1]_{\cos^2 \phi}\} \\
&\quad - \frac{1}{\rho_0} \frac{\partial}{\partial z} \{\rho_0 [(u' w')_1]\},
\end{aligned}
\]  

(S3.2)

where

\[ (u' v')_e \equiv u' v' - u' \bar{v} \iota. \]  

(S3.3)

Note that the linear terms are the same as those in the zonal wind equation for all waves. If we multiply \( \rho_0 u'_i \) to Eq. (S3.2), \( \rho_0 v'_i \) to the corresponding meridional
wind equation, and $\rho_0 / N^2 \cdot [\Phi_z ']'$, to the corresponding equation for the temperature,
then summing them and taking a zonal average, we obtain an energy conservation
equation for the wave components with a frequency range of “1”.

The wave energy conservation equation for the frequency range of “1” then becomes

$$\frac{d}{dt} (K_1' + P_1') = \{K, K_1'\} + \{\tilde{P}, P_1'\} + W_1 + D_1 + \Delta_1,$$  \hspace{1cm} (S3.4)

where the terms are the same as those in Eq.(2.5) except that all eddies (e.g., $u'$) are
replaced by eddies with the spectrum range of “1” (e.g., $u_1'$). The term $\Delta_1$ represents
the energy input from wave components with different frequency ranges, and if their
integrand is written as $\varepsilon(\Delta_1)$, their explicit form is written as

$$\varepsilon(\Delta_1) = \tilde{\varepsilon}(\Delta_1),$$  \hspace{1cm} (S3.5)

Where $\tilde{\varepsilon}(\Delta_1)$ is given by
\[
\varsigma(\Delta_t) = \frac{\rho_n u'_i}{\cos \phi} (u_u \frac{\partial u'_i}{\partial \lambda} - [u' \frac{\partial u'_i}{\partial \lambda}],_t) + \frac{\rho_n u'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - \frac{\partial u'_i}{\partial \phi} + \frac{\partial \vec{w}}{\partial \phi} - \frac{\partial u'_i}{\partial \phi}),_t \]

\[
- \frac{\rho_n u'_i}{a} \tan \phi \left( [u' v' + u'_i v'] - ([u' v'],_t) + \frac{\rho_n u'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{2 \rho_n u'_i}{a} \tan \phi \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n u'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n [\Phi_z]_t}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n [\Phi_z]_t}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n [\Phi_z]_t}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n [\Phi_z]_t}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n \mu'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n \mu'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n \mu'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n \mu'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n \mu'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n \mu'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n \mu'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n \mu'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n \mu'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n \mu'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n \mu'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n \mu'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n \mu'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n \mu'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n \mu'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n \mu'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right) + \frac{\rho_n v'_i}{a} \left( [u' v'] - ([u' v'],_t) + \frac{\rho_n v'_i}{a} (v_u \frac{\partial u'_i}{\partial \phi} - v' \frac{\partial u'_i}{\partial \phi}),_t \right)

(S3.6)
<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
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<tr>
<td>SURF</td>
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</tr>
<tr>
<td>ECONV</td>
<td>$2.984 \times 10^{14}$ W</td>
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<tr>
<td>DIAB</td>
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<td>FRICT</td>
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</tr>
<tr>
<td>SUM</td>
<td>$-0.035 \times 10^{14}$ W</td>
</tr>
</tbody>
</table>

**Table S4.** Climatological winter mean (December to March) of the total wave energy balance in the extratropical region (north of 45°N and below 100 hPa). “SURF” represents the energy input from the surface region, “ECONV” the energy conversion from the zonal-mean field, “DIAB” the diabatic heating, “FRICT” the frictional forcing, and “SUM” the summation of all forcings.