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Naked Singularities in the Vaidya Spacetime

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We investigate the property of a singularity which is formed by the spherical collapse of pure radiation from past null infinity, in terms of the congruence of null geodesics in the Vaidya spacetime. When the initial increase of mass due to the inflow of radiation is not so fast, the so-called shell-focusing singularity appears at the center. We present a class of examples in which such a singularity becomes globally naked. This is a counterexample to the cosmic censorship hypothesis.

§ 1. Introduction

In general relativity the appearance of a spacetime singularity is known to be inevitable under some physical conditions.[1] In order to avoid a disastrous breakdown of the predictability of the theory, the cosmic censorship hypothesis has been widely accepted in spite of the failure of its general proof. Namely, one claims that any gravitational collapse leads to such a singularity as that hidden within the black hole.

However, there exist some dynamical processes generating naked singularity. One of the examples is a shell-crossing singularity.[2] Under a suitable condition, dust shells cross together to generate the density singularity outside the event horizon on the way of their collapse towards the center. This naked singular shell does not cause any serious problem, since it is possible to determine the motion of the matter field through the singularity by treating it as a δ-functional distribution in the spacetime curvature.[3]

Another example is the shell-focusing singularity. When a spherically symmetric dust cloud collapse to the center sufficiently slowly, a piece of past-null singularity is formed at the center. This singularity always violates the predictability at least locally. When the collapse occurs sufficiently slowly for all the times, this singularity is visible from infinity. Unlike the shell-crossing singularity, this singularity cannot be handled by defining it as a δ-functional distribution.[4] In this sense, the spacetime containing the shell-focusing singularity is the strongest presently known counterexample to the cosmic censorship hypothesis.

The shell-focusing singularity is first discovered in examining the Tolman-Bondi spacetime which describes the spherical collapse of the dust cloud.[4] However, as the general solution of the Tolman-Bondi spacetime is very complex, it is troublesome to examine the properties of the singularities. Hiscock et al. found another example which possesses the shell-focusing singularity.[5] This is due to a special choice of the imploding Vaidya spacetime. These examples seem to show that any matter collapse will form the shell-focusing singularity under suitable conditions. As the Vaidya metric has a very simple form, it is possible to examine the conditions to generate the naked singularity and the property of the singularity by studying this spacetime.

The purpose of this paper is to examine the property of such a singularity in the general imploding Vaidya spacetime. When the radiation concentrates at the rate less than some critical value, the shell-focusing singularity is formed at the center. It appears at the first instance when the radiation hits the center. Whether this singularity is globally naked or not depends on the rate of the mass concentration in subsequent collapse.

In §2, we survey the structure of the imploding Vaidya spacetime and give the way of analyzing the property of such a naked singularity in terms of the congruence of null geodesics. In §3, we examine the local property of the singularities in the imploding Vaidya spacetime. Especially, the local property of the shell-focusing singularity is examined. In §4, we examine the global structure of the spacetime with the shell-focusing singularity through the search of the examples with the globally naked singularities. Section 5 is devoted to discussion and remarks.

§ 2. The imploding Vaidya spacetime

Let us consider the imploding Vaidya spacetime which describes the collapse of the pure radiation. The metric is given by [6]

$$ds^2=\left(1-\frac{2M(v)}{r}\right)dv^2-2dvdr-r^2d\Omega^2\,, \tag{2·1}$$

where $d\Omega^2$ is the metric of the unit two-sphere and mass $M(v)$ is an arbitrary function of the advanced time v. The energy-momentum tensor is given by

$$T_{\mu\nu}=\frac{\dot{M}}{4\pi r^2}k_\mu k_\nu\,, \tag{2·2}$$

where a dot denotes differentiation with respect to v. The null vector $k_\mu(=\delta_\mu{}^v)$ represents pure radiation (dust like null fluid) ingoing from infinity at the rate $\dot{M}$. The surface $r=2M(v)$ is the apparent horizon which is the boundary of the trapped region. If the condition $\dot{M}>0$ is satisfied, the event horizon always exists outside the apparent horizon.[1]

We are interested in the collapse of the pure radiation, so we assume that $M(v)=0$ for $v\leqq 0$ and $M(v)$ is a continuous and nondecreasing function for $v\geqq 0$. The curvature invariant is

$$R^{\lambda\mu\nu\sigma}R_{\lambda\mu\nu\sigma}=\frac{48M^2(v)}{r^6}\,. \tag{2·3}$$

The only candidate for the naked singularity is then the point $r=0$, $v=0$. This point is a singularity in the sense that Eq. (2·3) is divergent for the limit $r=0$ and $v\to +0$.

It is clear from Eq. (2·1) that the ingoing null geodesics are the lines $v=$constant. The outgoing null geodesics obey the equation

$$\frac{dr}{dv}=\frac{1}{2}\left(1-\frac{2M(v)}{r}\right). \tag{2·4}$$

We can see the structure of the spacetime by examining Eq. (2·4). For example, if there exists a solution of Eq. (2·4) which starts from the singularity and ends at the future null infinity, the singularity is globally naked. Unfortunately, we cannot solve Eq. (2·4) explicitly unless the special choice of $M(v)$ is given.

To examine the property of the singularity, we can analyze Eq. (2·4) in the following way. Suppose one special solution of Eq. (2·4) which passes $r=0$ at $v=0$ to be $r=r_0(v)$. This means that this spacetime has a singularity which is at least locally naked (i.e., visible from a neighboring future point). Then $r_0(v)$ must be related to the mass as

$$M(v)=\frac{1}{2}r_0(v)(1-2\dot{r}_0(v)). \tag{2·5}$$

If a function $r_0(v)$ is given, we can see what kind of spacetime with mass $M(v)$ admits the naked singularity.

To know the outgoing null geodesics near the special one $r_0(v)$, let us transform the variable r to z which is related to r as $r=r_0(v)+z$. Then Eq. (2·4) is reduced to

$$(r_0(v)+z)\frac{dz}{dv}+z\left(\dot{r}_0(v)-\frac{1}{2}\right)=0\,. \tag{2·6}$$

We can see the property of the naked singularity by analyzing Eq. (2·6) for small z.

§ 3. The local property of singularities

The only candidate for the naked singularity is the point[*] $r=0$, $v=0$ where the pure radiation concentrates for the first time as we have mentioned in §2. To investigate the local property of naked singularity, we need only to examine the behavior of the outgoing null geodesics near the point $r=0$, $v=0$.

Suppose $r_0\sim\beta v^\alpha$ with $\alpha>0$, $\beta>0$ for small v. Then we can see the behavior for the mass from Eq. (2·5) as follows: i) For $\alpha>1$, we have $M(v)\sim\frac{1}{2}\beta v^\alpha$. ii) For $\alpha=1$, we have $M(v)\sim\beta(\frac{1}{2}-\beta)v$. iii) For $\alpha<1$, we have $M(v)\sim-\alpha\beta^2v^{2\alpha-1}$. Then we can see that for small v, the mass should be $M(v)\sim\mu v^n$ with $\mu>0$, $n\geqq1$ if there exists the naked singularity. For the case $n=1$, μ should be $0<\mu\leqq1/16$.

Let us examine the property of the singularity for the case $M(v)\sim\mu v^n$ for small v with $\mu>0$, $n>0$. In the case $n>1$, the solution of Eq. (2·6) for small v which passes the point $r=0$, $v=0$ is given by[**]

$$z\sim\frac{C}{v^n}\exp\left\{\frac{-v^{-(n-1)}}{4\mu(n-1)}\right\}, \tag{3·1}$$

where C is an arbitrary constant. This solution represents the congruence of the outgoing null geodesics from the point $r=0$, $v=0$. This congruence is degenerate to the geodesic $r=r_0$ (i.e., every outgoing null geodesics $r=r(v)$ in the congruence approach the geodesic $r=r_0(v)$ sufficiently fast in such a way that $\dot{r}\to\dot{r}_0$) as they approach the singularity. (See Fig. 1 for the behavior of the outgoing null geodesics from the naked singularity). Thus this singularity is just the shell-focusing singularity.

Next we examine the case $n=1$ (or $M\sim\mu v$, $0<\mu\leqq\frac{1}{16}$). If we take the special outgoing null geodesics as $r=r_0\sim\beta v$, the solution of Eq. (2·6) for small v is given by[***]

$$v\sim\frac{2}{1-4\beta}z+C|z|^{2\beta/(1-2\beta)} \tag{3·2}$$

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[*] This terminology is not necessarily true (see Fig. 3 etc.) if the spacetime is expressed by the Penrose diagram.

[**] This equation is given by substituting $r_0\sim2\mu v^n$ into Eq. (2·6) and assuming $r_0\gg|z|$.

[***] The metrics derived from these equations are given by Hiscock et al.[5] in another form.

for $\beta \neq \frac{1}{4}(\mu < \frac{1}{16})$, and

$$v \sim 4z \ln|z| + Cz \tag{3·3}$$

for $\beta = \frac{1}{4}(\mu = \frac{1}{16})$, where C is an arbitrary constant. As μ is related to β by the equation $\mu = \beta(\frac{1}{2} - \beta)$, there are two outgoing null geodesics with the form $r \sim \beta v$ for a given μ, where

$$\beta = \beta_{\pm} = \frac{1 \pm \sqrt{1 - 16\mu}}{4}. \tag{3·4}$$

From Eq. (3·2), it is easy to see that the congruence of the outgoing null geodesics is degenerate to the geodesic $r \sim \beta_{-} v$ for $v \to 0$ and the geodesic $r \sim \beta_{+} v$ corresponds to the Cauchy horizon. In the case $\mu = \frac{1}{16}$ these two geodesics coincide. It is both the Cauchy horizon and the surface to which the congruence is degenerate. Thus the point $r=0$, $v=0$ is also the shell-focusing singularity.

As we have seen before, there is no naked singularity for the cases $n=1$, $\mu > \frac{1}{16}$ and $n<1$. What is the property of the singularity in these cases? For the case $n=1$, $\mu > \frac{1}{16}$, the solution of Eq. (2·4) for small v is given by[*)]

$$R \sim C \sqrt{\frac{x^2+1}{2\mu x^2 - x + 2}} \exp\left[\frac{1}{\sqrt{16\mu - 1}} \tan^{-1}\left\{\frac{4\mu}{\sqrt{16\mu - 1}}\left(x - \frac{1}{4\mu}\right)\right\}\right], \tag{3·5}$$

where $R = \sqrt{r^2 + v^2}$, $x = v/r$ and C is an arbitrary constant. We can easily see that there is no outgoing null geodesics from the point $R=0$ and all outgoing null geodesics go round the point $R=0$. Then this singularity is the end point of the usual spacelike singularity with $r=0$, $v>0$. For the case $n<1$, the property of the singularity is the same as that for the case $n=1$, $\mu > \frac{1}{16}$. From Eq. (2·4), it is clear that any outgoing null geodesics satisfy the condition $\dot{r} \leqq \frac{1}{2}$. They especially satisfy the condition $\dot{r} < 0$ in the trapped region ($r < 2M(v)$). As the radius of the apparent horizon $r_A (= 2M(v))$ grows very fast ($\dot{r}_A \to \infty$) at the point $r=0$, $v \to +0$, there cannot exist the outgoing null geodesics from the initial singularity. (See Fig. 2.)

§ 4. The globally naked singularity

In §3, we found that there appears the shell-focusing singularity which is at least locally naked when the mass is suitably chosen. In this section we shall examine whether the shell-focusing singularity becomes globally naked or not. This will crucially depend on the evolution of the event horizon.

The Cauchy horizon (suppose that it exists at the surface $r = r_C(v)$) is generated by an outgoing null geodesic. Then the condition $\dot{r}_C \leqq \frac{1}{2}$ is always satisfied. On the other hand, the radius of the apparent horizon which exists inside the event horizon, can grow more rapidly than the speed of light, i.e., $\dot{r}_A > \frac{1}{2}$ if the rate of subsequent mass flow is sufficiently large. Then it is always possible for the shell-focusing singularity to make locally naked (Cauchy horizon exists inside the event horizon) by controlling the subsequent mass flow. Conversely, it will be possible for the shell-focusing singularity to become globally naked by making the subsequent mass flow sufficiently small. Hereafter

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[*)] This equation is equivalent to that derived by Hiscock's metric.[5)] we shall give the examples of spacetime with a globally naked singularity and examine the structure of the spacetime.

To begin with, let us consider the case $n>1$ for small v. In order to obtain a global behavior of the null geodesic congruence, we must assume an exact form of $r_0(v)$. A typical example can be given by the form

$$r_0=\beta w^n \tag{4·1}$$

where $\beta>0$, $n>1$ and w is related to v in such a way as

$$v=2\beta w^n+w\,. \tag{4·2}$$

Then from Eq. (2·6) we have the mass

$$M(v)=\frac{\beta w^n}{2(1+2n\beta w^{n-1})}\,. \tag{4·3}$$

The mass satisfies the condition $\dot{M}>0$ for any positive v. The behavior of mass and the special outgoing null geodesic $r=r_0(v)$ is as follows: i) For sufficiently small v, $M(v)\sim(\beta/2)v^n$ and $r_0\sim\beta v^n$. ii) For sufficiently large v, $M(v)\sim(1/4n)(v/2\beta)^{1/n}$ and $r_0\sim v/2$. The behavior for small v guarantees that there exists the shell-focusing singularity at $r=0$, $v=0$. The behavior for large v means that the special outgoing null geodesic $r=r_0(v)$ goes to the future null infinity. Then this spacetime possesses the globally naked singularity. The spacetime structure is schematically shown in Fig. 1. In this case, the solution of Eq. (2·6), which satisfies the condition that $z\to 0$ as $v\to 0$, can be written as

$$z\sim C\exp\left\{-\frac{w^{-(n-1)}}{2\beta(n-1)}\right\} \tag{4·4}$$

for small v, where C is an arbitrary constant. Apparently, the geodesics with the limits $C\to+\infty$ and $-\infty$ correspond to the Cauchy horizon and the outgoing null geodesic which falls instantaneously to the singularity, respectively. An outgoing null geodesic corresponding to some negative C is the event horizon. We cannot, however, know the explicit value of C corresponding to the event horizon without solving Eq. (2·6) exactly. To see the spacetime structure explicitly, let us examine the case $n=2$ for which we can solve Eq. (2·6) exactly. (See the Appendix). The solution for $z>0$ is given by

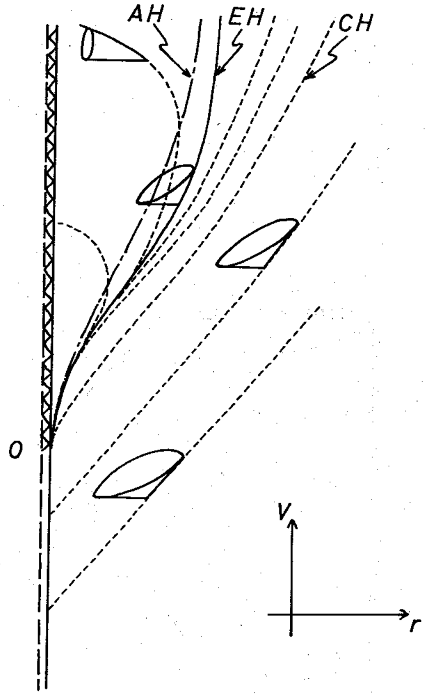

Fig. 1. Finkelstein diagram ($(r,\ v)$ plane) of the Vaidya spacetime with the mass $M(v)=0$ for $v\leqq 0$, $M(v)\sim\mu v^n(n>1)$ for small positive v and $\dot{M}\to 0$ for $v\to\infty$. The broken lines indicate the outgoing null geodesics. The dashed line indicates the point $r=0$. The jagged line is the singularity. The apparent horizon, event horizon and Cauchy horizon are represented by ***AH***, ***EH*** and ***CH*** respectively. The point ***O*** represents the shell-focusing singularity.

$$w=\sqrt{\frac{z}{\beta}}\cdot\frac{AJ_1(4\sqrt{\beta z})-N_1(4\sqrt{\beta z})}{AJ_0(4\sqrt{\beta z})-N_0(4\sqrt{\beta z})}, \tag{4·5}$$

where J_ν and N_ν are the Bessel functions of the first and second kinds, respectively, with order ν and A is an arbitrary constant. For $z<0$, the solution should be

$$w=-\sqrt{\frac{-z}{\beta}}\cdot\frac{BI_1(4\sqrt{-\beta z})-K_1(4\sqrt{-\beta z})}{BI_0(4\sqrt{-\beta z})+K_0(4\sqrt{-\beta z})}, \tag{4·6}$$

where I_ν and K_ν are the modified Bessel functions of the first and second kinds, respectively, with order ν and B is an arbitrary constant. We can see that the surface generated by the geodesic with the limit $A\to\infty$ corresponds to the Cauchy horizon. Similarly, the surface $B=0$ is the event horizon, because the outgoing null geodesics for $B>0$ turn to the singularity and those for $B<0$ escape to the future null infinity. Comparing Eqs. (4·5) and (4·6) with Eq. (4·4), we have

$$C=\frac{1}{4\beta}e^{\pi A-2\gamma} \tag{4·7}$$

for $z>0$ and

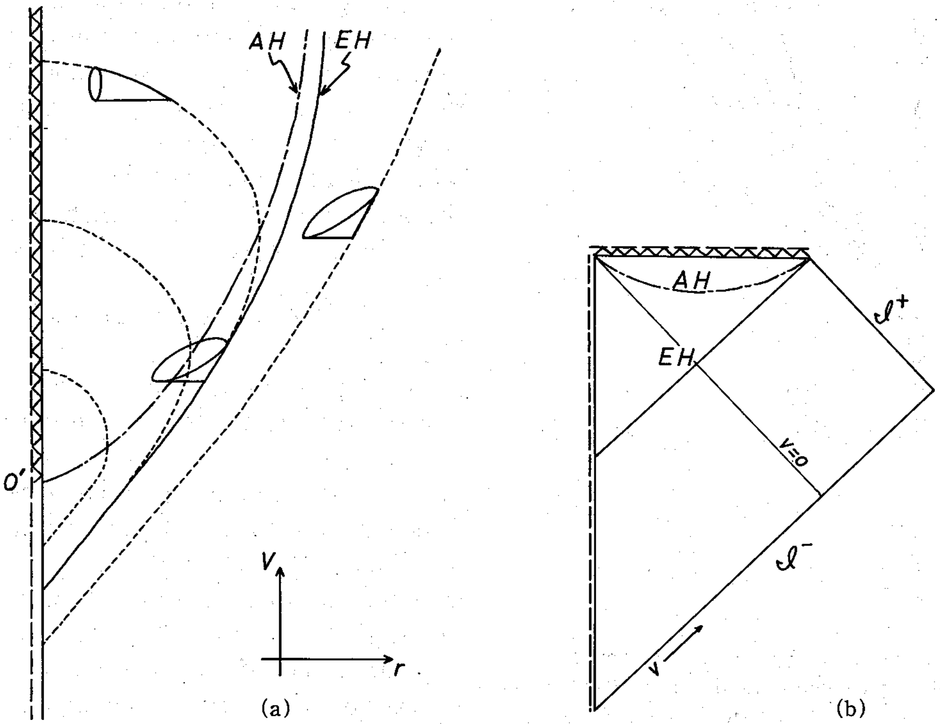

Fig. 2. Spacetime diagram of the Vaidya spacetime with the mass $M(v)=0$ for $v\leq 0$, $M(v)\sim\mu v^n$ ($n<1$ or $n=1, \mu>1/16$) for small positive v and $\dot{M}\to 0$ for $v\to\infty$. The broken lines indicate the outgoing null geodesics. The dashed line indicates the point $r=0$. The jagged line is the singularity. The apparent horizon and event horizon are represented by **AH** and **EH** respectively. The point **O'** represents the edge of the usual spacelike singularity. (a) Finkelstein diagram. (b) Penrose diagram. The null infinities are labeled by $\mathscr{I}^+$(future null) and $\mathscr{I}^-$(past null).

$$C=-\frac{1}{4\beta}e^{2B-2\gamma} \tag{4·8}$$

for $z<0$, where γ is Eluer's constant. This special model represents a typical feature of the spacetime with the globally naked singularity, which is schematically written in Fig. 1.

For the case $n=1$, an example that the shell-focusing singularity is globally naked is given by assuming that $M(v)=\mu v$ for any positive v. For this case, Eqs. (3·2) and (3·3) hold exactly. Then we can find that the geodesics $r=\beta_+ v$ and $r=\beta_- v$ correspond to the Cauchy horizon and the event horizon respectively in the case $\mu<1/16$. For the case $\mu=1/16$, they coincide, i.e., the geodesic $r=v/4$ corresponds to the Cauchy horizon and also the event horizon. Then this singularity is marginally naked. To make this singularity globally naked, we may stop the mass flow for some positive advanced time (say, $v=v_0$). After $v=v_0$ the spacetime becomes the Schwarzschild one, and so the event horizon exists on the surface $r=v_0/8$. As the Cauchy horizon exists on the surface $r=v_0/4$ at the advanced time $v=v_0$, it can reach to the future null infinity for the limit $v\to\infty$.

For the cases $n=1$, $\mu>1/16$ and $n<1$, we have seen that this spacetime cannot have the naked singularity. This will imply that the event horizon appears at some negative value of v and the singularity is always hidden inside the event horizon as is shown in Fig. 2.

The Penrose diagram for the case with globally naked (locally naked) singularity is schematically shown in Fig. 3 (Fig. 4).

We summarize the results as follows: If the mass is chosen to be $M(v)=0$ for $v\leqq 0$ and $M(v)\sim\mu v^n$ ($\mu>0$, $n>1$ or $0<\mu\leqq 1/16$, $n=1$) for small positive v, the shell-focusing singularity appears at $v=0$. By controlling the subsequent inflows of mass after the formation of the shell-focusing singularities, we can have the globally naked singularities. For the case $\dot{M}(0)>1/16$, no naked singular-

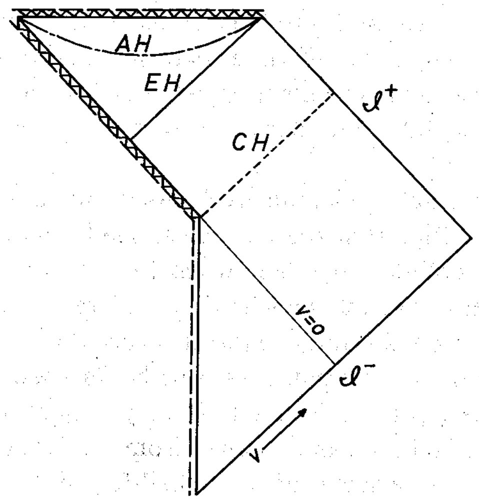

Fig. 3. Penrose diagram of the Vaidya spacetime with the globally naked shell-focusing singularity. The dashed line indicates the point $r=0$. The jagged line is the singularity. The apparent horizon, event horizon and Cauchy horizon are represented by ***AH***, ***EH***, ***CH*** respectively. The null infinities are labeled by $\mathscr{I}^+$(future null) and $\mathscr{I}^-$(past null).

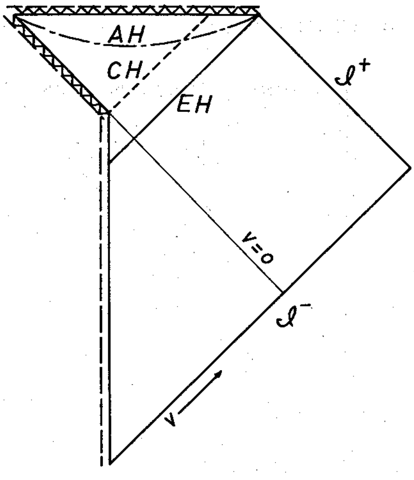

Fig. 4. The same as Fig. 3 for the case with the locally naked shell-focusing singularity.

ity can exist.

§ 5. Discussion and remarks

In the previous sections we have always assumed that the event horizon exists in the spacetime. This assumption will be always valid for any realistic case where the total collapsed masses are bounded. Furthermore, we have seen that the event horizon exists even if the total masses are not bounded (see Eq. (4·3)). On the other hand, from Eq. (3·5) we note that the spacetime with the mass $M(v)=\mu v(\mu>1/16)$ cannot have the event horizon and the future null infinity (any outgoing null geodesics are inevitably trapped to the singularity). What is the condition for the existence of the event horizon (and then future null infinity)? On the assumption that $M\geqq 0$ for $v\geqq 0$ and $M=0$ for $v\leqq 0$, the critical condition will become

$$\lim_{v\to\infty}\frac{M(v)}{v}=\frac{1}{16}, \tag{5·1}$$

as will be shown in the following: If the condition $\lim_{v\to\infty}(M/v)<1/16$ holds, we can istroduce a new Vaidya spacetime corresponding to the mass $\bar{M}$ which is defined by $\bar{M}\equiv v/16+D$ for $v\geqq -16D$ and $\bar{M}=0$ for $v<-16D$. Here D is a sufficiently large constant. Apparently the spacetime corresponding to the mass $\bar{M}$ has the event horizon as we have seen in §4. Because $\bar{M}$ is larger than M for all positive v, Eq. (2·4) implies that the velocity dr/dv of outgoing null geodesics is larger for M than for $\bar{M}$. Hence the spacetime corresponding to the mass M also can have the event horizon. On the other hand, if the condition $\lim_{v\to\infty}(M/v)>1/16$ holds, we can consider the mass $\bar{M}$ which is defined by $\bar{M}\equiv\mu v-D(1/16<\mu<\lim(M/v))$ for $v\geqq D/\mu$ and $\bar{M}\equiv 0$ for $v<D/\mu$. The spacetime corresponding to the mass $\bar{M}$ does not have the event horizon as we have seen in §4. Because M is larger than $\bar{M}$ for all positive v, the spacetime corresponding to the mass M also cannot have the event horizon by the same argument as before. We cannot say nothing about the case where Eq. (5·1) is valid. The structure of the spacetime without the event horizon is written in Fig. 5.

How can we understand the appearance of the naked singularity? As we have seen, the shell-focusing singularity appears on the condition that the innermost shell implodes and collides together, when the matter collapses with the sufficiently small rate. The situation is similar to the dust case though its detail is different. As Birkhoff's theorem states, gravity does not act on the innermost shell, in both cases. Then from the fact that the appearance of the shell-focusing singularity depends only on the rate of the inflow mass, it seems that the gradient of the gravity to the shell is important to the appearance of the shell-focusing singularity. When we consider the usual fluid, unlike the dust case, the pressure of the fluid will work to separate the shells and to make the

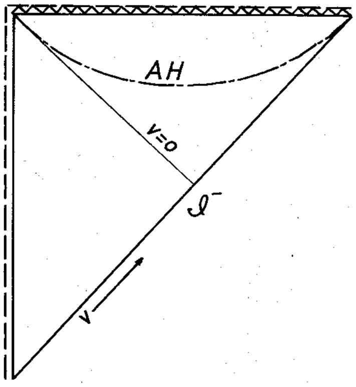

Fig. 5. The same as Fig. 3 for the case without shell-focusing singularity and event horizon.

gradient of the mass small as well as to suppress the implosion of the shells. Then the shell may form the shell-focusing singularity for some cases but not for another. However, in the case of the pure photon, this mechanism will not work, because the pressure cannot suppress the implosion of the shells. Then the imploding pure photon will also form the shell-focusing singularity as Vaidya's case. It seems that the distortion from the spherical symmetry cannot also save the fact.

If the appearance of the naked singularity is not avoided, how can we do about the predictability of the physics? One viewpoint is that the quantum effects in curved spacetime may prevent the formation of the naked singularity. Various works for calculating the particle creation in the spacetime with the naked singularity have been published.[7] The work by Hiscock et al.[5] is one of them. He calculated the energy-momentum tensor in the Vaidya spacetime with the mass $M(v)=\mu v$, and found that it becomes infinitely large for most cases. This means that the back-reaction may prevent the formation of the naked singularity. Whether this scheme is available for all the naked singularity in the general Vaidya spacetime is left to future work.

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Appendix

We shall derive Eqs. (4·5) and (4·6). Substituting Eqs. (4·1) and (4·2) with $n=2$ into Eq. (2·6), we have

$$(\beta w^2+z)\frac{dz}{dw}-\frac{z}{2}=0\,. \tag{A·1}$$

If we change the variable w to $y(=1/w)$, we can rewrite Eq. (A·1) as follows:

$$\frac{dy}{dz}+2y^2=-\frac{2\beta}{z}\,. \tag{A·2}$$

This is the Riccati equation. Putting y as

$$y=\frac{s}{t} \tag{A·3}$$

and transforming Eq. (A·2) to the linear equations of the vector $(s,\ t)$, we have

$$\frac{d}{dz}\begin{pmatrix} s \\ t \end{pmatrix}=\begin{pmatrix} 0 & -2\beta/z \\ 2 & 0 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix}, \tag{A·4}$$

which is reduced to the equations

$$\frac{d^2t}{dz^2}+\frac{4\beta}{z}t=0 \tag{A·5}$$

and

$$s=\frac{1}{2}\frac{dt}{dz}. \tag{A·6}$$

Then the general solution of t for positive z can be written as

$$t=C\sqrt{z}J_1(4\sqrt{\beta z})+D\sqrt{z}N_1(4\sqrt{\beta z}), \tag{A·7}$$

where C and D are arbitrary constants. From Eq. (A·6) s is easily derived. Then we obtain the final result, i.e., Eq. (4·5). Equation (4·6) is similarly obtained.

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