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LETTERS

On the Casimir Invariant of Hamiltonian Fluid Mechanics

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We investigate a singular Poisson bracket which appears in Hamiltonian fluid mechanics. It is shown that we can get the most general Casimir for such systems by solving the functional equations which correspond to zero eigenvalue equations. Especially, in the case of a three-dimensional adiabatic perfect fluid, the most general Casimir is shown to be the generalized enstrophy only.

[Hamiltonian, fluid mechanics, singular poisson bracket, adiabatic fluid, potential vorticity, Casimr, generalized enstrophy.]

Recently, Hamiltonian fluid mechanics, which is the application of the Hamiltonian field theory to fluid mechanics, has been developed.[1] One of the most useful applications in such a theory is to analyze the stability of the fluid system.[2] This method is called the energy-Casimir convexity method, and it is important to know the Hamiltonian structure, especially the Casimir invariant, of the given system to apply this method. In this sense, it is important to know what the most general Casimir of the given system is.

In the case of the Hamiltonian system with finite degrees of freedom, we can determine how many Casimirs exist in the given system simply by considering coranks of the Poisson matrix. This theorem is shown by Littlejohn.[3] However, in the case of the Hamiltonian system with infinite degrees of freedom, such as the fluid system, the dimension of the Poisson matrix becomes infinite, and then we cannot define the corank of such a matrix.

The purpose of this paper is to show how we can expand the Littlejohn theorem to the field theory and obtain the most general Casimir of the system. It is shown that the most general Casimir can be obtained only by solving the functional equations which correspond to zero eigenvalue equations.

The Poisson bracket which we are discussing is constructed by Eulerian fields only and has a noncanonical form. It can be obtained by projecting the canonical bracket of Lagrangian fields to the space made by Eulerian fields.[4] It has (A) bilinearity and anticommutativity and satisfies (B) the Leibniz formula and (C) Jacobi's identity which is inherited from the Lagrangian canonical bracket. Generally, we can denote such a bracket as

$$\{F, G\}=\iint \mathrm{d}x\,\mathrm{d}y J^{AB}(x, y)\frac{\delta F}{\delta\phi^A(x)}\frac{\delta G}{\delta\phi^B(y)}, \tag{1}$$

where F and G are functionals of the fields $\phi^A(x)$, $\delta F/\delta\phi^A(x)$ represents functional derivatives, suffices A, etc. represent species of the fields and repeated indices must be summed. Condition (A) requires

$$J^{AB}(x, y)=-J^{BA}(y, x), \tag{2}$$

and condition (C) requires

$$\int \mathrm{d}x \left(J^{AB}(x, u)\frac{\delta J^{CD}(y, z)}{\delta\phi^A(x)}+J^{AC}(x, y)\frac{\delta J^{DB}(z, u)}{\delta\phi^A(x)}+J^{AD}(x, z)\frac{\delta J^{BC}(u, y)}{\delta\phi^A(x)}\right)=0. \tag{3}$$

Condition (B) is automatically satisfied owning to the form (1).

The Casimir C is defined as the functional which satisfies

$$\{F, C\}=0, \tag{4}$$

for *any* functional F. Then it must satisfy the equation

$$\int \mathrm{d}y J^{AB}(x, y)\frac{\delta C}{\delta\phi^B(y)}=0. \tag{5}$$

To construct the Casimir C, let us consider the equation

$$\int \mathrm{d}y J^{AB}(x, y)\beta_B(y)=0. \tag{6}$$

This equation corresponds to the zero eigenvalue equation for the Poisson matrix. To construct the Casimir, let us consider the variational equation

$$\Omega^\alpha \equiv \int \mathrm{d}y \beta^\alpha_B(y)\delta\phi^B(y)=0, \tag{7}$$

where we have denoted suffix α to distinguish independent solutions $\beta_B(y)$ of eq. (6). These equations can be integrated, as we will show in the following.

As is shown in the Appendix, we can extend the usual theory of differential forms and the Frobenius theorem to the field theory. Note that in the field theory, variation plays the role of a differential. Now, let us consider the vector fields

$$X^A(x)=\int \mathrm{d}y J^{AB}(x, y)\frac{\delta}{\delta\phi^B(y)}. \tag{8}$$

They are orthogonal to the Pfaffian form (7). In fact, from eq. (6), we have

$$\langle\Omega^\alpha, X^A(x)\rangle=\int \mathrm{d}y J^{AB}(x, y)\beta^\alpha_B(y)=0. \tag{9}$$

Then the vectors (8) are systems of the generator of the Pfaffian form (7). We can show that these vectors are also complete because

$$[X^A(x), X^C(z)]=\int \mathrm{d}u C^{AC}_D(x, z, u)X^D(u), \tag{10}$$

with

$$C^{AC}_D(x, z, u)=\frac{\delta J^{AC}(x, z)}{\delta\phi^D(u)}, \tag{11}$$

are satisfied owning to eq. (3). Then, by the Frobenius theorem, eq. (7) is integrable. It also requires that there exist one forms θ^α_β which satisfies

$$\delta\Omega^\alpha=\theta^\alpha_\beta\wedge\Omega^\beta. \tag{12}$$

However, for $\delta\Omega^\alpha$, we have

$$\delta\Omega^\alpha=\iint \mathrm{d}x\,\mathrm{d}y\frac{\delta\beta^\alpha_B(y)}{\delta\phi^A(x)}\delta\phi^A(x)\wedge\delta\phi^B(y). \tag{13}$$

Unless $\delta\beta^\alpha_B(y)/\delta\phi^A(x)$ can be expressed as the product of functions of x and y, we cannot write $\delta\Omega^\alpha$ as eq. (12). However, the solutions $\beta^\alpha_B(y)$ are constructed by the fields $\phi^C(y)$ and their derivatives, and then $\delta\beta^\alpha_B(y)/\delta\phi^A(x)$ must contain $\delta(x-y)$ and their derivatives. Thus, we cannot rewrite them to the products of functions of x and y. The only method to harmonize these facts is to demand θ^α_β to vanish. This means eq. (7) is directly integrable.

Next, let us consider the concrete example of a three-dimensional adiabatic perfect fluid. In this system, the dynamical fields are velocity u^i ($i=1, 2, 3$), density ρ and entropy (per unit mass) S. The independent Poisson matrices are given by[4,5]

$$J^{ij}(x, y)=\frac{1}{\rho}\left(\frac{\partial u^j}{\partial x^i}-\frac{\partial u^i}{\partial x^j}\right)\delta(x-y), \tag{14}$$

$$J^{i(\rho)}(x, y)=-\frac{\partial}{\partial x^i}\delta(x-y), \tag{15}$$

$$J^{i(S)}(x, y)=\frac{1}{\rho}\frac{\partial S}{\partial x^i}\delta(x-y), \tag{16}$$

$$J^{(\rho)(S)}(x, y)=0. \tag{17}$$

Then eq. (6) for this case is given by

$$\frac{1}{\rho}(\partial_i u^j-\partial_j u^i)\beta_j-\partial_i\beta_{(\rho)}+\frac{1}{\rho}\partial_i S\cdot\beta_{(S)}=0, \tag{18}$$

$$\partial_i\beta_i=0, \tag{19}$$

$$\partial_i S\cdot\beta_i=0, \tag{20}$$

where we have denoted β_B as $(\beta_i, \beta_{(\rho)}, \beta_{(S)})$. From eqs. (19) and (20), β_i can be expressed by

$$\beta_i=\varepsilon_{ijk}\,\partial_j f\,\partial_k S, \tag{21}$$

where f is some function. If we use this relation in eq. (18), we have

$$\frac{1}{\rho}(\varepsilon_{ijk}\,\partial_j u_k\,\partial_i f+\beta_{(S)})\,\mathrm{d}S-q\,\mathrm{d}f-\mathrm{d}\beta_{(\rho)}=0, \tag{22}$$

where q is the potential vorticity and defined by

$$q \equiv \frac{1}{\rho} \varepsilon_{ijk} \partial_j u_k \partial_i S. \tag{23}$$

If we define functions γ and α by

$$\rho\gamma \equiv \varepsilon_{ijk} \partial_j u_k \partial_i f + \beta_{(S)}, \tag{24}$$

$$\alpha \equiv \beta_{(\rho)} + qf, \tag{25}$$

eq. (22) can be rewritten by

$$\mathrm{d}\alpha = f\,\mathrm{d}q + \gamma\,\mathrm{d}S. \tag{26}$$

It is easy to see that the most general solution for eq. (26) is given by

$$\alpha = F(q, S),$$
$$f = F_q(q, S),$$
$$\gamma = F_S(q, S), \tag{27}$$

where F is an arbitrary function of q and S, and F_q represents $\partial F/\partial q$, etc. From these solutions, we have

$$\beta_i = \varepsilon_{ijk} \partial_j F_q \partial_k S,$$
$$\beta_{(\rho)} = F - qF_q,$$
$$\beta_{(S)} = -\varepsilon_{ijk} \partial_j u_k \partial_i F_q + \rho F_S. \tag{28}$$

Then, it is easy to integrate eq. (7), and we have

$$\Omega = \delta \int \mathrm{d}^3 x \rho F(q, S). \tag{29}$$

Then the most general Casimir for this system is $\int \mathrm{d}^3 x \rho F(q, S)$ only. Note that we can still always generalize any Casimir C as $G(C)$ with G being an arbitrary function.

Now, let us consider the perfect fluid without entropy. The Poisson matrices are the same as the case with entropy except that the entropy components are absent. However, in this case, eqs. (6) and (7) give the most general Casimir with the form

$$\int \mathrm{d}^3 x (c_1 \varepsilon_{ijk} u_i \partial_j u_k + c_2 \rho), \tag{30}$$

where c_1 and c_2 are arbitrary constants. Then we have lost the arbitrary function in this case. In this sense, for the three-dimensional perfect fluid, whether entropy (or some quantity which conserves on fluid particles) exists or not is very crucial for Hamiltonian structure. In contrast with the three-dimensional case, it is now easy to see that a two-dimensional perfect fluid[6] has the general Casimir with an arbitrary function regardless of the existence of entropy.* However, the number of arbitrary functions is reduced when the entropy is absent.

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A Frobenius Theorem in the Field Theory

The Frobenius theorem[7] demands that the following system of Pfaffian equations ω ($i=1, 2, \cdots, n$ and $\alpha=1, 2, \cdots, s(<n)$)

$$\omega^\alpha = a_i^\alpha(x)\,\mathrm{d}x^i = 0, \tag{31}$$

can be integrated if (i) there exists one form θ_β^α satisfying the equation

$$\mathrm{d}\omega^\alpha = \theta_\beta^\alpha \wedge \omega^\beta, \tag{32}$$

or (ii) the system of generators $A(\omega)$ which are defined by

$$A(\omega) = \{X \mid \langle \omega^\alpha, X \rangle = 0,\ \alpha = 1, 2, \cdots, s\}, \tag{33}$$

is complete: i.e., there exists function C_{tu}^v which satisfies

$$[X_t, X_u] = C_{tu}^v X_v, \tag{34}$$

for any element $X_t, X_u \in A(\omega)$.

We can consider the field theory as the limit of the discrete theory. To do this, let us connect the numbers ϕ_i ($i=1, 2, \cdots, n$) and the function $\phi(x)$ by

$$\phi_i = \phi(x_i), \tag{35}$$

where $x_i = a + i\Delta$ and $\Delta = (b-a)/n$. Then, we can treat the variation of the function $\phi(x)$ as the limit of the set of differentials for the variables ϕ_i. For example, the functional derivative can be defined by

$$\frac{\delta a[\phi(\cdot)]}{\delta\phi(y)} = \lim_{n\to\infty} \frac{1}{\Delta} \frac{\partial a(\cdots, \phi_i, \cdots)}{\partial \phi_{i(y)}}, \tag{36}$$

* In the case with (without) entropy, the most general Casimir can be calculated as the three-dimensional case, and it is given by $\int \mathrm{d}^2 x h(pF(S)+K(S))$ ($\int \mathrm{d}^2 x h F(p)$) where h and p are the depth of water and the two-dimensional potential vorticity, respectively, and F and K are arbitrary functions.

where $i(y)$ represents the number which stisfies $x_{i(y)}=y$. In this way, we can treat a variational problem as the limit of a differential problem and then we can expand the theory of differential forms into field theories.[7,8] We shall denote the differential form by δ for field theories.

Now, let us consider one form,

$$\bar{\omega}^{\alpha}=\sum_{i=1}^{n} \Delta \cdot A_{i\beta}^{\alpha}(\cdots, \phi_j, \cdots)\, \mathrm{d}\phi_i^{\beta}. \tag{37}$$

The integrability conditions for the Pfaffian system $\bar{\omega}^{\alpha}=0$ are given by the usual Flobenius theorem. The one form (37) converges to

$$\omega^{\alpha}=\int_a^b \mathrm{d}x A_{\beta}^{\alpha}(x)[\phi(\cdot)]\,\delta\phi^{\beta}(x), \tag{38}$$

as n tends to infinity. Then, the integrability condition for $\omega^{\alpha}=0$ is given by the same condition as that for $\bar{\omega}^{\alpha}=0$. Condition (i) does not change the form, but we must change the differential to the variation. Condition (ii) also does not change the form, but we must expand the inner product of the form with the vector field for the field theory. We must define it as $\langle \delta\phi^{\alpha}(x), \delta/\delta\phi_{\beta}(y)\rangle=\delta_{\alpha\beta}\delta(x-y)$, if we consider eq. (36) and the correspondence of δ_{ij}/Δ to the delta function. Then the vector fields take the form

$$X^{t}=\int_a^b \mathrm{d}x B_{\beta}^{t}(x)[\phi(\cdot)]\frac{\delta}{\delta\phi_{\beta}(x)}, \tag{39}$$

and the orthogonal condition for ω^{α} takes

$$\langle \omega^{\alpha}, X^{t}\rangle=\int_a^b \mathrm{d}x A_{\beta}^{\alpha}(x) B_{\beta}^{t}(x)=0. \tag{40}$$

References

1) R. Salmon: Ann. Rev. Fluid Mech. **20** (1988) 225.
2) D. D. Holm, J. Marsden, T. Ratiu and A. Weinstein: Phys. Rep. **123** (1985) 1.
3) R. G. Littlejohn: Am. Inst. Phys. Conf. Proc. **88** (1982) 47.
4) Y. Kuroda: Fluid Dyn. Res. **5** (1990) 273.
5) P. J. Morrison and J. M. Greene: Phys. Rev. Lett. **45** (1980) 790; Addition **48** (1982) 569.
6) R. Salmon: J. Fluid Mech. **196** (1988) 345.
7) T. Kimura and R. Sugano: *Analytical Dynamics in Terms of Differential Forms* (McGraw-Hill, Tokyo, 1988) [in Japanese].
8) C. Crnkovic and E. Witten: In *300 years of gravitation*, ed. S. W. Hawking and W. Israel (Cambridge UP, 1987) p. 676.